2013 Missouri Collegiate Mathematics Competition
Session I

1. Consider the sets of consecutive integers $\{1\}$, $\{2,3\}$, $\{4,5,6\}$, $\{7,8,9,10\}$, $\ldots$, where each set contains one more element than the preceding one, and where the first element of each set is one more than the last element of the preceding set. Let S_n be the sum of the elements in the nth set. Find S_{32}.

Solution.

The sets have average values $1, 2.5, 5, 8.5, \ldots$. Since the differences are in arithmetic progression, the average value can be expressed as a quadratic function of n; it is not difficult to see that this quadratic function is $(n^2+1)/2$. This means that the sum of the nth set is $n(n^2+1)/2$, so S_{32} equals 16400.

(Suggested by 1967 AHSME, Problem 39.)

2. Find all positive real solutions x to the equation

$$4\lfloor x \rfloor = 3x\{x\}, \tag{1}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\{x\} = x - \lfloor x \rfloor$.

Solution.

Let $x > 0$ be a solution of (1). Write $x = k + \alpha$, where k is a nonnegative integers, and $0 \le \alpha < 1$. Then (1) is equivalent to

$$4k = 3\alpha(k+\alpha) \quad \text{or} \quad 3\alpha^2 + 3k\alpha - 4k = 0.$$

Solving for α gives

$$\alpha = \frac{-3k \pm \sqrt{9k^2+48k}}{6}. \tag{2}$$

The assumption that $\alpha \ge 0$ excludes the minus from the plus/minus. The assumption that $\alpha < 1$ is successively equivalent to

$$\begin{aligned} -3k + \sqrt{9k^2+48k} &< 6 \\ \sqrt{9k^2+48k} &< 3k+6 \\ 9k^2+48k &< 9k^2+36k+36 \\ 12k &< 36 \\ k &< 3. \end{aligned}$$

Hence, $k \in \{0,1,2\}$.

- If $k = 0$, then (2) implies that $\alpha = 0$, a contradiction since we are assuming $x > 0$.

- If $k = 1$, then $\alpha = \frac{-3+\sqrt{57}}{6}$, so that
$$x = \frac{1}{2} + \frac{\sqrt{57}}{6}.$$
- If $k = 2$, then $\alpha = \frac{-6+\sqrt{132}}{6} = \frac{-3+\sqrt{33}}{3}$, so that
$$x = 1 + \frac{\sqrt{33}}{3}.$$

3. Evaluate the following where $p > 1$.

$$\frac{1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots}.$$

Solution.

Since $p > 1$, we know that the numerator and denominator are p-series and therefore convergent. Let

$$\begin{aligned} x &= \frac{1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} \\ &= \frac{1 + \frac{1}{2^p}(2-1) + \frac{1}{3^p} + \frac{1}{4^p}(2-1) + \frac{1}{5^p} + \frac{1}{6^p}(2-1) + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + + \frac{1}{5^p} - \frac{1}{6^p} + \cdots} \\ &= \frac{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \frac{1}{5^p} - \frac{1}{6^p} + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \frac{1}{5^p} - \frac{1}{6^p} + \cdots} + \frac{2(\frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{6^p} + \cdots)}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \frac{1}{5^p} - \frac{1}{6^p} + \cdots} \\ &= 1 + \frac{\frac{2}{2^p}(1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots)}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \frac{1}{5^p} - \frac{1}{6^p} + \cdots} \\ &= 1 + \frac{2}{2^p}x. \end{aligned}$$

Therefore,

$$x = \frac{1}{1 - \frac{2}{2^p}}.$$

4. Neither *Mathematica* nor *Maple* can find the exact value of the following definite integral. Can you? We think you can. Do it!

$$\int_0^2 (3x^2 - 3x + 1)\cos(x^3 - 3x^2 + 4x - 2)\,dx.$$

Solution.

We first split up the integral as

$$\int_0^2 (3x^2 - 6x + 4)\cos(x^3 - 3x^2 + 4x - 2)\,dx + \int_0^2 (3x - 3)\cos(x^3 - 3x^2 + 4x - 2)\,dx,$$

If we set $u = x^3 - 3x^2 + 4x - 2$ in the first integral, we get $\int_{-2}^{2} \cos u\,du$ which equals $2\sin(2)$.

If we set $v = x - 1$ in the second integral, we get $\int_{-1}^{1} 3v\cos(v^3 + v)\,dv$. Applying the definition of even and odd function shows $\cos(v^3 + v)$ to be an even function, which means that $v\cos(v^3 + v)$ is an odd function and the second integral is 0.

Therefore, the original integral is equal to $2\sin(2)$.

(By Loren Larson, from an Iowa contest.)

5. Define the sequence $\{a_n\}_{n=1}^{\infty}$ by

$$a_1 = 1, \quad a_2 = 1, \quad a_3 = 2,$$

and for $n \geq 3$,

$$a_{n+1} = \frac{n a_n a_{n-2}}{a_{n-1}}.$$

(a) Prove that a_n is a positive integer for all $n \geq 1$.

(b) Define the sequence $\{b_n\}_{n=1}^{\infty}$ by

$$b_n = \frac{a_n}{\sqrt{(n+1)!}} \qquad \text{for} \quad n \geq 1.$$

Prove that $\{b_n\}$ is bounded.

Solution.

(a) For $n \geq 1$,

$$a_{n+3} = \frac{(n+2)a_{n+2}a_n}{a_{n+1}},$$

so that

$$a_{n+4} = \frac{(n+3)a_{n+3}a_{n+1}}{a_{n+2}} = \frac{(n+3)(n+2)a_{n+2}a_n a_{n+1}}{a_{n+1}a_{n+2}},$$

and so

$$a_{n+4} = (n+3)(n+2)a_n. \tag{3}$$

Since $a_4 = \frac{3a_3a_1}{a_2} = 6$, a_1, a_2, a_3, a_4 are positive integers. Assume that a_k is a positive integer for all $k < n+4$. By strong induction and (3), a_{n+4} is a positive integer.

(b) We have

$$\begin{aligned} b_{n+4} &= \frac{a_{n+4}}{\sqrt{(n+5)!}} \\ &= \frac{(n+3)(n+2)a_n}{\sqrt{(n+1)!(n+2)(n+3)(n+4)(n+5)}} \\ &= \frac{a_n}{\sqrt{(n+1)!}} \cdot \sqrt{\frac{(n+3)(n+2)}{(n+4)(n+5)}} \\ &\leq \frac{a_n}{\sqrt{(n+1)!}} \\ &= b_n. \end{aligned}$$

By induction, since $b_1, b_2, b_3, b_4 \leq 1$, $b_n \leq 1$ for all $n \geq 1$.

2013 Missouri Collegiate Mathematics Competition
Session II

1. Let r be a real number with $0 < r \leq 1$. Let $S(r)$ be the region in the (x, y)-plane bounded by the curve $y = x^r$, the x-axis, and the line $x = 1$. Let $G(r)$ denote the centroid of $S(r)$. Let ℓ_r be the line that passes through the origin and $G(r)$. Let $T(r)$ be the region in the (x, y)-plane bounded by ℓ_r, the x-axis, and the line $x = 1$. Find the value of r that minimizes the ratio of the area of $T(r)$ to the area of $S(r)$.

Solution.

We have

$$A = \text{area}(S(r)) = \int_0^1 x^r\,dx = \frac{1}{r+1}.$$

The coordinates of the centroid are given by

$$\begin{aligned}\overline{x} &= \frac{1}{A}\int_0^1 x \cdot x^r\,dx = \frac{1}{A}\cdot\frac{1}{r+2}\\ \overline{y} &= \frac{1}{A}\cdot\frac{1}{2}\int_0^1 (x^r)^2\,dx = \frac{1}{A}\cdot\frac{1}{2}\cdot\frac{1}{2r+1}.\end{aligned}$$

Thus, the equation of line ℓ_r is

$$y = \left(\frac{\overline{y}}{\overline{x}}\right)x = \left(\frac{r+2}{2(2r+1)}\right)x,$$

so that

$$\text{area}(T(r)) = \frac{r+2}{4(2r+1)}.$$

Let $f(r)$ denote the ratio we are trying to minimize. Then

$$\begin{aligned}f(r) &= \frac{\text{area}(T(r))}{\text{area}(S(r))}\\ &= \frac{(r+2)(r+1)}{4(2r+1)} = \frac{r^2+3r+2}{4(2r+1)}.\end{aligned}$$

Since $f(r)$ is continuous on $[0, 1]$, it has an absolute minimum on that interval, either at a critical point or at an endpoint of the interval. We have

$$\begin{aligned}f'(r) &= \frac{(2r+1)(2r+3)-(r^2+3r+2)(2)}{4(2r+1)^2}\\ &= \frac{2r^2+2r-1}{4(2r+1)^2}.\end{aligned}$$

For r in the interval $[0,1]$, $f(r)$ has a unique critical point; it occurs at the positive solution of the quadratic $2r^2+2r-1=0$. This value is

$$r_0 = \frac{-1+\sqrt{3}}{2}.$$

We note that $f'(r)<0$ for $r<r_0$ and $f'(r)>0$ for $r>r_0$. Hence, the minimum value of f occurs at r_0.

2. Let

$$f(r) = \sum_{j=2}^{2013} \frac{1}{j^r} = \frac{1}{2^r} + \frac{1}{3^r} + \cdots + \frac{1}{2013^r}.$$

Find

$$\sum_{k=2}^{\infty} f(k).$$

Solution.

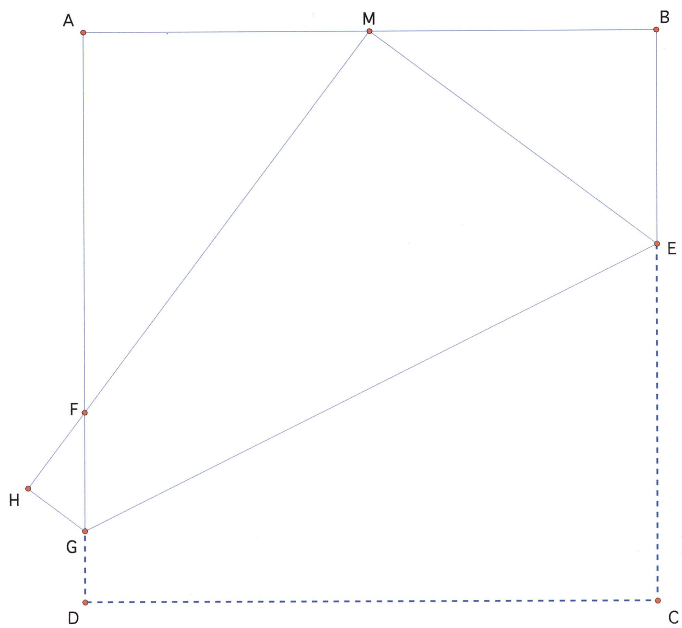
After switching the order of summation, we note that the inner sum is a convergent geometric series. Thus, we can proceed as follows:

$$\begin{aligned}\sum_{k=2}^{\infty}\sum_{j=2}^{2013}\frac{1}{j^k} &= \sum_{j=2}^{2013}\sum_{k=2}^{\infty}\frac{1}{j^k} = \sum_{j=2}^{2013}\frac{\frac{1}{j^2}}{1-\frac{1}{j}} \\ &= \sum_{j=2}^{2013}\frac{1}{j^2-j} = \sum_{j=2}^{2013}\frac{1}{j(j-1)} = \sum_{j=2}^{2013}\left(\frac{1}{j-1}-\frac{1}{j}\right) \\ &= \frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{2012}-\frac{1}{2013} = \frac{2012}{2013}.\end{aligned}$$

(Similar to a problem from the 2008 Harvard-MIT Mathematics Tournament.)

3. Define an *Egyptian triangle* to be one that is similar to a 3-4-5 right triangle. A square sheet of paper is folded so that one corner touches the midpoint of the opposite side. The folded sheet will thus be composed of three triangles "one sheet thick" and a quadrilateral "two sheets thick." Prove that all three triangles are Egyptian triangles.

Solution.



Label points as in the figure, and denote the measure of an angle or a segment by $| * |$. Because each triangle includes a corner of the square, all three are right triangles. For the same reason, $\angle EMF$ is a right angle, and so $\angle AMF$ and $\angle EMB$ are complementary. Because $\angle MEB$ and $\angle EMB$ are also complementary, we get that $|\angle MEB| = |\angle AMF|$. Similarly, $|\angle MFA| = |\angle EMB|$. This proves that triangle AMF and BEM are similar. Because they are vertical angles $|\angle AFM| = |\angle HFG|$, making $|\angle FGH| = |\angle FMA|$, and thus triangles FGH and FMA are similar. This proves that all three triangles are right triangles and are similar.

It suffices to prove that one of the triangles is similar to a 3-4-5 triangle. Let the length of a side of the square be $2s$. Then $|AM| = |MB| = s$. Because of the fold, $|CE| = |EM|$. Let this length be z, making $|EB| = 2s - z$. By the Pythagorean Theorem,

$$z^2 = s^2 + (2s - z)^2$$

and solving for z gives

$$z = \frac{5}{4}s.$$

Thus, the sides of triangle EBM are $\frac{3}{4}s$, s, and $\frac{5}{4}s$, making it similar to a 3-4-5 right triangle.

4. Prove that $\frac{x^2+y^2}{4} \leq e^{x+y-2}$ for $x \geq 0$ and $y \geq 0$.

Solution.

Consider the quotient $\frac{x^2+y^2}{4e^{x+y-2}} = \frac{e^2}{4}\left(\frac{x^2+y^2}{e^{x+y}}\right)$ and let $f(x,y) = \frac{x^2+y^2}{e^{x+y}}$. We will determine the extrema for f (and put the constant factor back in later).

$$\begin{aligned} f_x(x,y) &= \frac{2x - x^2 - y^2}{e^{x+y}} \\ f_y(x,y) &= \frac{2y - x^2 - y^2}{e^{x+y}}. \end{aligned}$$

Setting $f_x(x,y) = 0$ and $f_y(x,y) = 0$ we get the critical points to be $(0,0)$ and $(1,1)$. Checking the boundary, on $y = 0$,

$$f'(x,0) = \frac{2x - x^2}{e^x}$$

and on $x = 0$

$$f'(0,y) = \frac{2y - y^2}{e^y}$$

so the critical points on the boundary are $(0,0)$, $(2,0)$, and $(0,2)$. Evaluating at the critical points,

$$f(0,0) = 0, \qquad f(1,1) = 2e^{-2}, \qquad f(2,0) = f(0,2) = 4e^{-2}$$

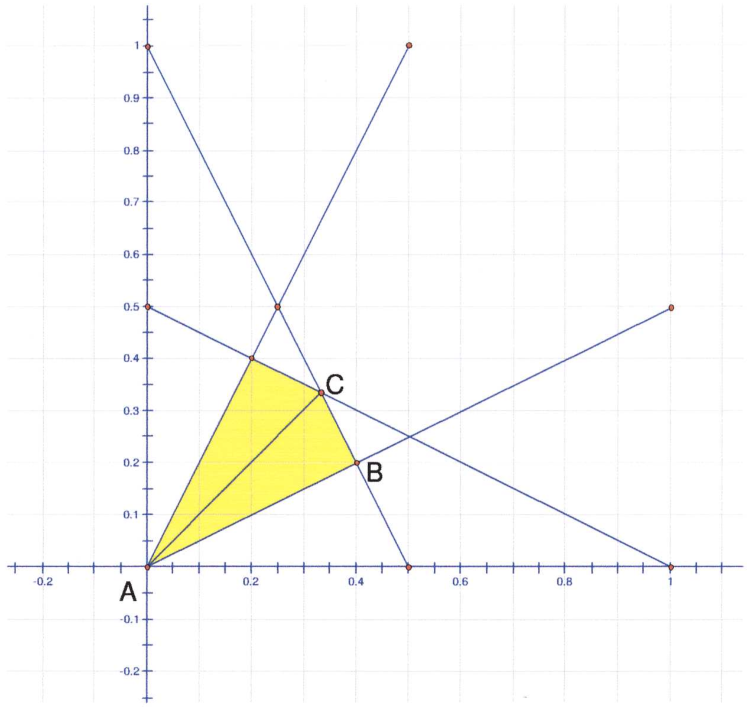
so, in the closed first quadrant, the maximum is $4e^{-2}$ and the minimum is 0. Therefore, for $x \geq 0$ and $y \geq 0$,

$$\begin{aligned} 0 &\leq f(x,y) \leq \frac{4}{e^2} \\ 0 &\leq \frac{e^2}{4} f(x,y) \leq 1 \\ 0 &\leq \frac{x^2+y^2}{4} \leq e^{x+y-2} \end{aligned}$$

as desired.

5. A point (x,y) is chosen at random from the square with vertices $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ (uniform distribution on the unit square). Find the probability that x and y are side lengths of an isosceles triangle of perimeter at most 1.

Solution.



The set of points (x, y) that correspond to side lengths of an isosceles triangle of perimeter at most 1 consists of the union of two overlapping regions in the square

- The first region, corresponding to side lengths x, x, y, is the triangle bounded by $y = 0$, $y = 2x$ (from the triangle inequality), and $2x + y = 1$ (from the perimeter requirement). The second, corresponding to side lengths x, y, y, is the triangle bounded by $x = 0$, $x = 2y$, and $x + 2y = 1$. These congruent triangles each have area $\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$.

- The intersection of the two regions is a convex quadrilateral which is symmetric about the line $y = x$. We find the area of the quadrilateral (and hence the desired probability) by computing the area of the portion below the line $y = x$ and doubling the result. The portion below the line $y = x$ is a right triangle ABC, with $A = (0, 0)$, and B and C to be determined. Point B is the intersection of $x = 2y$ and $2x + y = 1$. Thus, the coordinates of B are $\left(\frac{2}{5}, \frac{1}{5}\right)$. Point C is the intersection $y = x$ and $2x + y = 1$. Thus, the coordinates of C are $\left(\frac{1}{3}, \frac{1}{3}\right)$. From the distance formula, $AB = \frac{\sqrt{5}}{5}$ and $BC = \frac{\sqrt{5}}{15}$. Thus, the area of the intersection is $2\left(\frac{1}{2}\right)\left(\frac{\sqrt{5}}{5}\right)\left(\frac{\sqrt{5}}{15}\right) = \frac{1}{15}$.

- The desired probability is therefore
$$\frac{1}{8} + \frac{1}{8} - \frac{1}{15} = \frac{11}{60}.$$