2014 Missouri Collegiate Mathematics Competition
Session I

1. Let $0 \le a \le 1$ be given. Determine all nonnegative continuous functions f on $[0,1]$ (or prove there are none) which satisfy the following three conditions.

$$\int_0^1 f(x)\,dx = 1$$
$$\int_0^1 xf(x)\,dx = a$$
$$\int_0^1 x^2 f(x)\,dx = a^2.$$

2. Let T_0 be an isosceles triangle with base b and base angles α. Define a sequence $\{T_n\}_{n=0}^{\infty}$ of triangles, recursively, as follows. The base of each triangle T_n is b, and the base angles of triangle T_{n+1} have half the measure of the base angles of triangle T_n. Evaluate

$$\lim_{n\to\infty} \frac{2^n \cdot \text{area}(T_n)}{\text{area}(T_0)}.$$

3. Find the sum of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots$, where the terms are the reciprocals of the positive integers whose only prime factors are twos and threes.

4. Let Q be an arbitrary convex quadrilateral with vertices A_0, A_1, A_2, and A_3 ordered counterclockwise. Let M_i be the midpoint of side A_iA_{i+1} interpreting the subscripts mod 4. Then draw the medians A_iM_{i+2} (again interpreting the subscripts mod 4). Prove that the quadrilaterals $A_0M_0A_2M_2$ and $A_1M_1A_3M_3$ have equal areas.

5. For which positive integers a does there exist a right triangle with integer sides, at least one of which is a?

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Session II

1. 101 marbles are numbered from 1 to 101. The marbles are divided between two baskets A and B. The marble numbered 40 is in basket A. This marble is removed from basket A and put in basket B. The average of all the numbers on the marbles in A increases by $\frac{1}{4}$. The average of all the numbers on the marbles in B increases by $\frac{1}{4}$ too. How many marbles were there originally in basket A?

2. If $f(x) = \sin^6\left(\frac{x}{4}\right) + \cos^6\left(\frac{x}{4}\right)$, find $f^{(2014)}(0)$. (Note that $f^{(n)}(x)$ refers to the nth derivative of f evaluated at x.)

3. A polynomial $P(x)$ is known to be of the form $P(x) = x^{15} - 9x^{14} + \cdots - 7$, where the ellipsis $(\cdots)$ represent unknown intermediate terms. It is also known that all roots of $P(x)$ are integers. Find the roots (including multiplicities) of $P(x)$.

4. Find all real solutions to the equation $4x^2 - 40\lfloor x \rfloor + 51 = 0$. (Note that $\lfloor x \rfloor$ denotes the floor function, the greatest integer less than or equal to x.)

5. Given the fact that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$, evaluate the integral

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+(y-x)^2+y^2)}\,dx\,dy.$$