2014 Missouri Collegiate Mathematics Competition
Session I

1. Let $0 \leq a \leq 1$ be given. Determine all nonnegative continuous functions f on $[0,1]$ (or prove there are none) which satisfy the following three conditions.

$$\int_0^1 f(x)\,dx = 1$$
$$\int_0^1 xf(x)\,dx = a$$
$$\int_0^1 x^2 f(x)\,dx = a^2.$$

Solution.

There is no such function. Multiply the three equations by a^2, $-2a$, and 1, respectively, and add, getting

$$\int_0^1 f(x)(a-x)^2\,dx = 0.$$

But the only nonnegative continuous function satisfying this condition is $f(x) \equiv 0$, and so no continuous nonnegative function can satisfy the three conditions.

2. Let T_0 be an isosceles triangle with base b and base angles α. Define a sequence $\{T_n\}_{n=0}^{\infty}$ of triangles, recursively, as follows. The base of each triangle T_n is b, and the base angles of triangle T_{n+1} have half the measure of the base angles of triangle T_n. Evaluate

$$\lim_{n\to\infty} \frac{2^n \cdot \text{area}(T_n)}{\text{area}(T_0)}.$$

Solution.

The base angles of triangle T_n have measure $\frac{\alpha}{2^n}$. Thus, the altitude, h_n of triangle T_n is given by $\tan\left(\frac{\alpha}{2^n}\right) = \frac{h_n}{b/2}$, so that $h_n = \frac{1}{2}b\tan\left(\frac{\alpha}{2^n}\right)$, and $\text{area}(T_n) = \frac{1}{4}b^2\tan\left(\frac{\alpha}{2^n}\right)$. Therefore,

$$\frac{2^n \cdot \frac{1}{4}b^2\tan\left(\frac{\alpha}{2^n}\right)}{\frac{1}{4}b^2\tan\alpha} = \frac{2^n \cdot \tan\left(\frac{\alpha}{2^n}\right)}{\tan\alpha} = \frac{\alpha\cdot\left(\tan\left(\frac{\alpha}{2^n}\right)/\left(\frac{\alpha}{2^n}\right)\right)}{\tan\alpha},$$

so that

$$\lim_{n\to\infty} \frac{2^n \cdot \text{area}(T_n)}{\text{area}(T_0)} = \frac{\alpha}{\tan\alpha}.$$

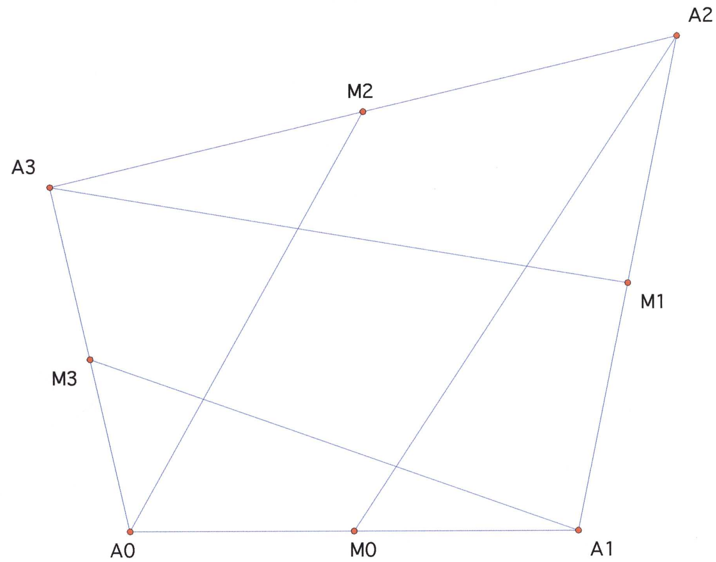
3. Find the sum of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots$, where the terms are the reciprocals of the positive integers whose only prime factors are twos and threes.

Solution.

$$
\begin{aligned}
&1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{2}\frac{1}{2} + \frac{1}{3}\frac{1}{3} + \frac{1}{2}\frac{1}{2}\frac{1}{3} + \cdots \\
&= \left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1\left(\frac{1}{3}\right)^0 + \left(\frac{1}{2}\right)^0\left(\frac{1}{3}\right)^1 + \left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^1\left(\frac{1}{3}\right)^1 \\
&\quad + \left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^1 + \cdots \\
&= \left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right)\left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots\right) \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^m\left(\frac{1}{3}\right)^n = \sum_{m=0}^{\infty}\left(\frac{1}{2}\right)^m\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n \\
&= \left(\frac{1}{1-\frac{1}{2}}\right)\left(\frac{1}{1-\frac{1}{3}}\right) = 3.
\end{aligned}
$$

4. Let Q be an arbitrary convex quadrilateral with vertices A_0, A_1, A_2, and A_3 ordered counterclockwise. Let M_i be the midpoint of side A_iA_{i+1} interpreting the subscripts mod 4. Then draw the medians A_iM_{i+2} (again interpreting the subscripts mod 4). Prove that the quadrilaterals $A_0M_0A_2M_2$ and $A_1M_1A_3M_3$ have equal areas.

Solution.



Draw diagonal A_0A_2. Then because M_0 is the midpoint of A_0A_1 the triangles $A_0M_0A_2$ and $M_0A_1A_2$ have equal bases and the same altitude, and so have equal areas. Similarly, triangles $A_2M_2A_0$ and $M_2A_3A_0$ have equal areas. Therefore, denoting the area of the figure by $|*|$, $|A_0M_0A_2M_2| = |M_0A_1A_2| + |M_2A_3A_0|$ making $|A_0M_0A_2M_2| = \frac{1}{2}|Q|$. Draw diagonal A_1A_3 and use the same reasoning to show that $|A_1M_1A_3M_3| = \frac{1}{2}|Q|$, making $|A_0M_0A_2M_2| = |A_1M_1A_3M_3|$.

From Rick Mabry, *Crosscut convex quadrilaterals*, Math. Mag., **84** (2011), 16–25.

5. For which positive integers a does there exist a right triangle with integer sides, at least one of which is a?

Solution.

If there were such a triangle for $a = 1$, then by the Pythagorean Theorem, there would be positive integers b, c such that either $b^2 + c^2 = 1$ or $1 = c^2 - b^2 = (c + b)(c - b)$, neither of which is possible. Likewise, for $a = 2$, neither of the equations $b^2 + c^2 = 4$ nor $4 = c^2 - b^2$ has positive integer solutions.

For all positive integers $a \geq 3$, however, there does exist such a triangle with a as a leg. To see this, we observe first that if the positive integer a_0 is a leg of a right triangle and k is any positive integer, then ka_0 is a leg of a right triangle since $a_0^2 + b_0^2 = c_0^2$ implies that $(ka_0)^2 + (kb_0)^2 = (kc_0)^2$. Since every positive integer greater than or equal to three is a multiple either of an odd number or of 4, it suffices to consider the cases $a = 4$ and $a \geq 3$ odd. The Pythagorean triple $(3, 4, 5)$ dispenses with the case $a = 4$. Suppose now that $a \geq 3$ is odd. We write the Pythagorean Theorem in the form $a^2 = (c + b)(c - b)$. To find positive integers b and c for which this last equation holds, we solve the system

$$\begin{aligned} c + b &= a^2 \\ c - b &= 1, \end{aligned}$$

obtaining

$$b = \frac{a^2 - 1}{2}, \quad c = \frac{a^2 + 1}{2}.$$

Since a is odd, these are integers; since $a \geq 3$, they are positive.

2014 Missouri Collegiate Mathematics Competition
Session II

1. 101 marbles are numbered from 1 to 101. The marbles are divided between two baskets A and B. The marble numbered 40 is in basket A. This marble is removed from basket A and put in basket B. The average of all the numbers on the marbles in A increases by $\frac{1}{4}$. The average of all the numbers on the marbles in B increases by $\frac{1}{4}$ too. How many marbles were there originally in basket A?

Solution.

Let $x_0, x_1, \ldots, x_n$ (respectively $x_{n+1}, \ldots, x_{100}$) denote the numbers on the marbles originally in basket A (respectively in basket B) with $x_0 = 40$. The original averages are

$$m_A = \frac{40 + x_1 + \cdots + x_n}{n+1} \quad \text{and} \quad m_B = \frac{x_{n+1} + \cdots + x_{100}}{100 - n}.$$

After the marble numbered 40 is removed from basket A and put in basket B, the averages become

$$m'_A = \frac{x_1 + \cdots + x_n}{n} \quad \text{and} \quad m'_B = \frac{40 + x_{n+1} + \cdots + x_{100}}{101 - n}.$$

Expressing $m'_A = m_A + \frac{1}{4}$ and $m'_B = m_B + \frac{1}{4}$, we easily get:

$$\frac{x_1 + \cdots + x_n}{n(n+1)} = \frac{40}{n+1} + \frac{1}{4}$$

and

$$\frac{x_{n+1} + \cdots + x_{100}}{(100 - n)(101 - n)} = \frac{40}{101 - n} - \frac{1}{4}$$

which yield

$$x_1 + \cdots + x_n = \frac{n^2 + 161n}{4} \qquad (1)$$

and

$$x_{n+1} + \cdots + x_{100} = \frac{5900 + 41n - n^2}{4}. \qquad (2)$$

Since $x_1 + \cdots + x_n + x_{n+1} + \cdots + x_{100} = (1 + 2 + \cdots + 100 + 101) - 40 = \frac{101 \times 102}{2} - 40$, we obtain by addition of (**??**) and (**??**):

$$5900 + 202n = 204 \times 101 - 160.$$

Hence, $n = 72$ and there were initially 73 marbles in basket A.

This is Problem 3 from the Dutch Mathematical Olympiad - 1999. The solution is by Michel Bataille.

2. If $f(x) = \sin^6\left(\frac{x}{4}\right) + \cos^6\left(\frac{x}{4}\right)$, find $f^{(2014)}(0)$. (Note that $f^{(n)}(x)$ refers to the nth derivative of f evaluated at x.)

Solution.

We have

$$\begin{aligned}
&\sin^6\left(\frac{x}{4}\right) + \cos^6\left(\frac{x}{4}\right)\\
&= \left(\sin^2\left(\frac{x}{4}\right) + \cos^2\left(\frac{x}{4}\right)\right)\left(\sin^4\left(\frac{x}{4}\right) - \sin^2\left(\frac{x}{4}\right)\cos^2\left(\frac{x}{4}\right) + \cos^4\left(\frac{x}{4}\right)\right)\\
&= \sin^4\left(\frac{x}{4}\right) + 2\sin^2\left(\frac{x}{4}\right)\cos^2\left(\frac{x}{4}\right) + \cos^4\left(\frac{x}{4}\right) - 3\sin^2\left(\frac{x}{4}\right)\cos^2\left(\frac{x}{4}\right)\\
&= 1 - \frac{3}{4}\sin^2\left(\frac{x}{2}\right) = 1 - \frac{3}{4}\left(\frac{1-\cos x}{2}\right)\\
&= \frac{5}{8} + \frac{3}{8}\cos x.
\end{aligned}$$

Thus, $f^{(2014)}(x) = -\frac{3}{8}\cos x$ and $f^{(2014}(0) = -\frac{3}{8}$.

Similar to a problem from the 2008 Harvard-MIT Mathematics Tournament.

3. A polynomial $P(x)$ is known to be of the form $P(x) = x^{15} - 9x^{14} + \cdots - 7$, where the ellipsis $(\cdots)$ represent unknown intermediate terms. It is also known that all roots of $P(x)$ are integers. Find the roots (including multiplicities) of $P(x)$.

Solution.

Let $r_1, r_2, \ldots, r_{15}$ be the roots of $P(x)$. Since $P(x) = \prod_{i=1}^{15}(x - r_i)$, we can see that the sum of the roots is 9 and the product is 7. Since all of the roots are assumed to be integers, the product tells us that one of the roots is ± 7 and all of the others are ± 1. Note that -7 cannot be a root since the sum could not then be any larger than 7. Therefore, one of the roots is 7. In order for the sum to be 9, eight of the other roots are 1 and the remaining six are -1.

From a U and I contest.

4. Find all real solutions to the equation $4x^2 - 40\lfloor x\rfloor + 51 = 0$. (Note that $\lfloor x\rfloor$ denotes the floor function, the greatest integer less than or equal to x.)

Solution.

Let $n = \lfloor x\rfloor$ and $\alpha = x - n$; then $0 \le \alpha < 1$. Substituting for x, we obtain $4(n+\alpha)^2 - 40n + 51 = 0$, from which the quadratic formula gives $\alpha = -n \pm \sqrt{10n - \frac{51}{4}}$. A necessary condition for real solutions is $n \ge 2$. From the restrictions on α, we see that there is no need to consider the negative square root, and we also get two inequalities: firstly, $-n+\sqrt{10n-\frac{51}{4}} \geq 0$ (equivalently, $n^2-10n+\frac{51}{4} \leq 0$), which gives $1.5 \leq n \leq 8.5$; secondly, $-n+\sqrt{10n-\frac{51}{4}} < 1$ (equivalently, $n^2-8n+\frac{55}{4} > 0$), which gives $n < 2.5$ or $n > 5.5$. Therefore, the only values of n that need to be considered are $n = 2, 6, 7$, or 8.

Substituting these four values into $x = n+\alpha = \sqrt{10n-\frac{51}{4}} = \frac{\sqrt{40n-51}}{2}$, we find that the solutions are

$$x = \frac{\sqrt{29}}{2}, \quad \frac{3\sqrt{21}}{2}, \quad \frac{\sqrt{229}}{2}, \quad \text{and} \quad \frac{\sqrt{269}}{2}.$$

From the 1999 Canada National Olympiad.

5. Given the fact that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$, evaluate the integral

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+(y-x)^2+y^2)}\,dx\,dy.$$

Solution.

Make the change of variables $u = x+y$, $v = -x+y$ and note that $u^2+v^2 = 2x^2+2y^2$. Also, $x = \frac{1}{2}(u-v)$ and $y = \frac{1}{2}(u+v)$, making the Jacobian of the transformation: $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$. Thus,

$$\begin{aligned}
I &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2} e^{-(\frac{1}{2}u^2+\frac{3}{2}v^2)}\,du\,dv \\
&= \frac{1}{2}\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\,du\right)\left(\int_{-\infty}^{\infty} e^{-\frac{3}{2}v^2}\,dv\right) \\
&= \frac{1}{2}\left(\sqrt{2}\int_{-\infty}^{\infty} e^{-w^2}\,dw\right)\left(\frac{\sqrt{2}}{\sqrt{3}}\int_{-\infty}^{\infty} e^{-z^2}\,dz\right) \\
&= \frac{\pi}{\sqrt{3}}.
\end{aligned}$$