On Identities of Ruggles, Horadam, and Howard

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June 27, 2016

Outline

Let $\{F_n\}$ and $\{L_n\}$ be the Fibonacci and Lucas numbers, respectively.

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That is, $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ and $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

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Ruggles proved that for $n \geq 0$ and $k \geq 1$,

$$F_{n+2k} = L_k F_{n+k} + (-1)^{k+1} F_n.$$

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$$F_{n+2k} = L_k F_{n+k} + (-1)^{k+1} F_n.$$

Horadam generalized this result to a general second order recurrence relation.

Theorem

Let w_0, w_1, a, and $b \neq 0$ be integers. Define

$$w_n = aw_{n-1} + bw_{n-2}$$

for $n \geq 2$. In addition, define $x_0 = 2$, $x_1 = a$, and for $n \geq 2$,

$$x_n = ax_{n-1} + bx_{n-2}.$$

Then for integers $n \geq 0$ and $k \geq 1$,

$$w_{n+2k} = x_k w_{n+k} + (-1)^{k+1} b^k w_n.$$

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Finally, we will state and prove a similar result for a general fourth order recurrence.

Outline

Before we can prove our results, we need a general definition and a general theorem.

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Definition

Let $r \geq 2$ be an integer. Let w_0, w_1, ..., w_{r-1}, and p_1, p_2, ..., $p_r \neq 0$ be integers. Define

$$w_n = p_1 w_{n-1} + p_2 w_{n-2} + \cdots + p_r w_{n-r} \text{ for } n \geq r.$$

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Definition

Let $r \geq 2$ be an integer. Let w_0, w_1, ..., w_{r-1}, and p_1, p_2, ..., $p_r \neq 0$ be integers. Define

$$w_n = p_1 w_{n-1} + p_2 w_{n-2} + \cdots + p_r w_{n-r} \text{ for } n \geq r.$$

We now give our theorem.

Theorem

Let $k \geq 1$ be an integer. Let M be the $r \times r$ matrix given by

$$\begin{pmatrix} p_1 & p_2 & p_3 & \cdots & p_{r-1} & p_r \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Let

$$p(x) = \det(xI - M^k) = \sum_{i=0}^{r} C_k(i,r)x^i$$

be the characteristic polynomial of M^k.

Theorem

Then

$$\sum_{i=0}^{r} C_k(i,r) w_{n+ik} = 0.$$

Here is the proof.

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By the Cayley-Hamilton Theorem, every matrix satisfies its characteristic polynomial. Therefore,

$$p(M^k) = \det(xI - M^k) = \sum_{i=0}^{r} C_k(i,r)(M^k)^i = 0. \qquad (1)$$

Multiplying both sides of (1) on the right by

$$\begin{pmatrix} w_n \\ w_{n-1} \\ \vdots \\ w_{n-r+1} \end{pmatrix}$$

gives

$$\sum_{i=0}^{r} C_k(i,r) M^{ik} \begin{pmatrix} w_n \\ w_{n-1} \\ \vdots \\ w_{n-r+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2}$$

But, it can be shown, by induction on m, that

$$\begin{pmatrix} p_1 & p_2 & p_3 & \cdots & p_{r-1} & p_r \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}^m \begin{pmatrix} w_n \\ w_{n-1} \\ \vdots \\ w_{n-r+1} \end{pmatrix} = \begin{pmatrix} w_{n+m} \\ w_{n+m-1} \\ \vdots \\ w_{n+m-r+1} \end{pmatrix}. \tag{3}$$

Letting $m = ik$ in (3) and substituting the right-hand side of (3) into (2), we obtain

$$\sum_{i=0}^{r} C_k(i,r) \begin{pmatrix} w_{n+ik} \\ w_{n+ik-1} \\ \vdots \\ w_{n+ik-r+1} \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{r} C_k(i,r) w_{n+ik} \\ \sum_{i=0}^{r} C_k(i,r) w_{n+ik-1} \\ \vdots \\ \sum_{i=0}^{r} C_k(i,r) w_{n+ik-r+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4}$$

Letting $m = ik$ in (3) and substituting the right-hand side of (3) into (2), we obtain

$$\sum_{i=0}^{r} C_k(i,r) \begin{pmatrix} w_{n+ik} \\ w_{n+ik-1} \\ \vdots \\ w_{n+ik-r+1} \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{r} C_k(i,r) w_{n+ik} \\ \sum_{i=0}^{r} C_k(i,r) w_{n+ik-1} \\ \vdots \\ \sum_{i=0}^{r} C_k(i,r) w_{n+ik-r+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4}$$

Equating the first component of the two column vectors of (4) gives the result.

Since the leading coefficient of the characteristic polynomial of M^k is 1, we have $C_k(r, r) = 1$.

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Therefore, we can rewrite the statement of the theorem as

$$w_{n+rk} = -C_k(r-1, r)w_{n+(r-1)k} - C_k(r-2, r)w_{n+(r-2)k} - \cdots - C_k(0, r)w_n.$$

We wish to investigate the case $r = 3$. That is, we want to obtain the sequences $-C_k(2,3)$, $-C_k(1,3)$, and $-C_k(0,3)$. This will give us an identity for the general third order recurrence.

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Likewise, we will study the case $r = 4$. To find an identity for the general fourth order recurrence, we will determine the sequences $-C_k(3,4)$, $-C_k(2,4)$, $-C_k(1,4)$, and $-C_k(0,4)$.

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Likewise, we will study the case $r = 4$. To find an identity for the general fourth order recurrence, we will determine the sequences $-C_k(3,4)$, $-C_k(2,4)$, $-C_k(1,4)$, and $-C_k(0,4)$.

To demonstrate the use of the Theorem, we will give an example to find an identity for the Tribonacci sequence. In this example, we let $r = 3$ and $p_1 = p_2 = p_3 = 1$.

Definition

Let $T_0 = 0$, $T_1 = 0$, and $T_2 = 1$. Define

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \text{ for } n \geq 3.$$

The polynomials producing $-C_k(2,3)$, $-C_k(1,3)$, and $-C_k(0,3)$ are the following.

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$$\det(xI - I) = \det \begin{pmatrix} x-1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{pmatrix} = x^3 - 3x^2 + 3x - 1.$$

$$\det(xI - M) = \det \begin{pmatrix} x-1 & -1 & -1 \\ -1 & x & 0 \\ 0 & -1 & x \end{pmatrix} = x^3 - x^2 - x - 1.$$

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$$\det(xI - M^2) = \det \begin{pmatrix} x-2 & -2 & -1 \\ -1 & x-1 & -1 \\ -1 & 0 & x \end{pmatrix} = x^3 - 3x^2 - x - 1.$$

$$\det(xI - M^3) = \det\begin{pmatrix} x-4 & -3 & -2 \\ -2 & x-2 & -1 \\ -1 & -1 & x-1 \end{pmatrix} = x^3 - 7x^2 + 5x - 1.$$

$$\det(xI - M^3) = \det\begin{pmatrix} x-4 & -3 & -2 \\ -2 & x-2 & -1 \\ -1 & -1 & x-1 \end{pmatrix} = x^3 - 7x^2 + 5x - 1.$$

$$\det(xI - M^4) = \det\begin{pmatrix} x-7 & -6 & -4 \\ -4 & x-3 & -2 \\ -2 & -2 & x-1 \end{pmatrix} = x^3 - 11x^2 - 5x - 1.$$

$$\det(xI - M^5) = \det\begin{pmatrix} x-13 & -11 & -7 \\ -7 & x-6 & -4 \\ -4 & -3 & x-2 \end{pmatrix} = x^3 - 21x^2 - x - 1.$$

$$\det(xI - M^5) = \det\begin{pmatrix} x-13 & -11 & -7 \\ -7 & x-6 & -4 \\ -4 & -3 & x-2 \end{pmatrix} = x^3 - 21x^2 - x - 1.$$

$$\det(xI - M^6) = \det\begin{pmatrix} x-24 & -20 & -13 \\ -13 & x-11 & -7 \\ -7 & -6 & x-4 \end{pmatrix} = x^3 - 39x^2 + 11x - 1.$$

Here are the beginning values of the sequences.

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Table: Values of Specific Third Order Sequences

k	0	1	2	3	4	5	6	7	8	9	10
T_k	0	0	1	1	2	4	7	13	24	44	81
$-C_k(2,3)$	3	1	3	7	11	21	39	71	131	241	443
$-C_k(1,3)$	-3	1	1	-5	5	1	-11	15	-3	-23	41
$-C_k(0,3)$	1	1	1	1	1	1	1	1	1	1	1

Definition

Let $a_0 = 3$, $a_1 = 1$, and $a_2 = 3$. Define

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} \text{ for } n \geq 3.$$

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Definition

Let $b_0 = -3$, $b_1 = 1$, and $b_2 = 1$. Define

$$b_n = -b_{n-1} - b_{n-2} + b_{n-3} \text{ for } n \geq 3.$$

We then have the following corollary.

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Corollary

Let $n \geq 0$ and $k \geq 1$. Then

$$T_{n+3k} = a_k T_{n+2k} + b_k T_{n+k} + T_n.$$

Outline

Definition

Let w_0, w_1, w_2, a, b, and $c \neq 0$ be integers. Define

$$w_n = aw_{n-1} + bw_{n-2} + cw_{n-3} \text{ for } n \geq 3.$$

Definition

Let $x_0 = 3$, $x_1 = a$, and $x_2 = a^2 + 2b$. Define

$$x_n = ax_{n-1} + bx_{n-2} + cx_{n-3}, \text{ for } n \geq 3.$$

Definition

Let $y_0 = -3$, $y_1 = b$, and $y_2 = 2ac - b^2$. Define

$$y_n = -by_{n-1} - acy_{n-2} + c^2 y_{n-3}, \text{ for } n \geq 3,$$

Here are some values of the $\{x_n\}$ and $\{y_n\}$ sequences.

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Table: Values of Third Order Sequences

n	0	1	2	3
x_n	3	a	a^2+2b	$a^3+3ab+3c$
y_n	-3	b	$2ac-b^2$	$-3abc+b^3-3c^2$

Table: Values of Third Order Sequences

n	4
x_n	$a^4 + 4a^2b + 4ac + 2b^2$
y_n	$-2a^2c^2 + 4ab^2c - b^4 + 4bc^2$

Table: Values of Third Order Sequences

n	4
x_n	$a^4 + 4a^2b + 4ac + 2b^2$
y_n	$-2a^2c^2 + 4ab^2c - b^4 + 4bc^2$

Table: Values of Third Order Sequences

n	5
x_n	$a^5 + 5a^3b + 5a^2c + 5ab^2 + 5bc$
y_n	$5a^2bc^2 - 5ab^3c + 5ac^3 + b^5 - 5b^2c^2$

Table: Values of Third Order Sequences

n	6
x_n	$a^6 + 6a^4b + 6a^3c + 9a^2b^2 + 12abc + 2b^3 + 3c^2$
y_n	$2a^3c^3 - 9a^2b^2c^2 + 6ab^4c - 12abc^3 - b^6 + 6b^3c^2 - 3c^4$

Theorem

Let $n \geq 0$ and $k \geq 1$ be integers. Then

$$w_{n+3k} = x_k w_{n+2k} + y_k w_{n+k} + c^k w_n.$$

Outline

The statement and proof of the fourth order result is long and technical. We will not go into the theorem's statement or proof here.

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However, we will state the result for the Tetranacci sequence.

The Tetranacci sequence is A000078 in the OEIS.

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Definition

Let $T_0 = 0$, $T_1 = 0$, $T_2 = 0$, and $T_3 = 1$. Define

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4} \text{ for } n \geq 4.$$

The next sequence is A073817 in the OEIS.

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Definition

Let $a_0 = 4$, $a_1 = 1$, $a_2 = 3$, and $a_3 = 7$. Define

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} \text{ for } n \geq 4.$$

The next sequence is A074193 in the OEIS.

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Definition

Let $b_0 = -6$, $b_1 = 1$, $b_2 = 3$, $b_3 = 1$, $b_4 = -17$, and $b_5 = 16$. Define

$$b_n = -b_{n-1} - 2b_{n-2} - 2b_{n-3} + 2b_{n-4} - b_{n-5} + b_{n-6} \text{ for } n \geq 6.$$

The last sequence sequence is A073937 in the OEIS.

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Definition

Let $c_0 = 4$, $c_1 = 1$, $c_2 = -1$, and $c_3 = 1$. Define

$$c_n = c_{n-1} - c_{n-2} + c_{n-3} + c_{n-4} \text{ for } n \geq 4.$$

Here are the beginning values of the sequences.

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Table: Values of Specific Fourth Order Sequences

k	0	1	2	3	4	5	6	7	8	9	10
T_k	0	0	0	1	1	2	4	8	15	29	56
a_k	4	1	3	7	15	26	51	99	191	367	708
b_k	-6	1	3	1	-17	16	15	-13	-81	127	58
c_k	4	1	-1	1	7	6	-1	1	15	19	4

Table: Values of Specific Fourth Order Sequences

k	11	12	13	14	15
T_k	108	208	401	773	1490
a_k	1365	2631	5071	9975	18842
b_k	-175	-329	885	31	-1424
c_k	1	31	53	27	6

Corollary

Let $n \geq 0$ and $k \geq 1$. Then

$$T_{n+4k} = a_k T_{n+3k} + b_k T_{n+2k} + c_k T_{n+k} + (-1)^{k+1} T_n.$$

References

1. A. F. Horadam, *Basic properties of a certain generalized sequence of numbers*, **3.3** (1965), 161–176.
2. F. T. Howard, *A tribonacci identity*, The Fibonacci Quarterly, **39.4** (2001), 352–357.
3. OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, **http://oeis.org**.
4. S. Rabinowitz, *Algorithmic manipulation of third-order linear recurrences*, The Fibonacci Quarterly, **34.5** (1996), 447–464.
5. I. D. Ruggles, *Some Fibonacci results using Fibonacci-type sequences*, The Fibonacci Quarterly, **1.2** (1963), 75–80.
6. A. G. Shannon and A. F. Horadam, *Some properties of third-order recurrence relations*, The Fibonacci Quarterly, 10.2 (1972), 135–145.

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