The 48th Mersenne Prime, GIMPS, the LL Test, and Perfect Numbers

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University of Central Missouri

April 22, 2013

1. **Mersenne Primes**
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     - $2^{31} - 1$ is prime
5. **Two Theorems**

Prime Numbers

- A **prime number** is an integer, greater than 1, which has exactly two factors, itself and one.

Prime Numbers

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- Prime Numbers Less Than 100:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41,$$
$$43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$$

Mersenne Numbers

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- Examples of Mersenne numbers are:

$$
\begin{aligned}
3 &= 2^2 - 1 \\
7 &= 2^3 - 1 \\
31 &= 2^5 - 1 \\
127 &= 2^7 - 1 \\
2047 &= 2^{11} - 1
\end{aligned}
$$

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\end{aligned}
$$

- $2047 = 2^{11} - 1 = 23 \times 89.$

Marin Mersenne

- Mersenne primes are named after a 17th-century French monk and mathematician

Marin Mersenne (1588-1648)

News About 48th Mersenne Prime

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- Pronunciation of M57885161
  http://www.isthe.com/chongo/tech/math/digit/m57885161/prime-d.html

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- George Woltman founded GIMPS in January 1996 and wrote the prime testing software.
- Scott Kurowski wrote the PrimeNet server that supports GIMPS. In 1997 he founded Entropia, a distributed computing software company.

GIMPS

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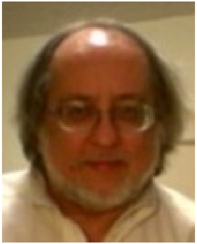
- As of February 5, 2013, GIMPS had a sustained throughput of approximately 129 trillion floating-point operations per second).
- The GIMPS project consists of 98,980 users, 574 teams, and 730,562 CPUs.
- UCM has over 1000 computers performing LL-tests on Mersenne numbers.

GIMPS People

Woltman

Kurowski



Crandall

GIMPS Links

- The GIMPS home page can be found at: http://www.mersenne.org

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- A Mersenne Prime discussion forum can be found at: http://www.mersenneforum.org

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Lucas-Lehmer Test

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Definition

Let $S_1 = 4$ and

$$S_{n+1} = S_n^2 - 2 \text{ for } n \geq 1.$$

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Definition

Let $S_1 = 4$ and

$$S_{n+1} = S_n^2 - 2 \text{ for } n \geq 1.$$

- The first few terms of the S sequence are:

$4,\ 14,\ 194,\ 37634,\ 1416317954,\ 2005956546822746114,$
$4023861667741036022825635656102100994, \ldots$

Lucas-Lehmer Test

Let p be a prime number. Then

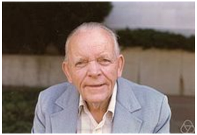
$$M_p = 2^p - 1 \text{ is prime}$$

$$\text{if and only if}$$

$$S_{p-1} \bmod M_p = 0.$$

Lucas-Lehmer Test

Lucas



Lehmer

Theorem

$M_{11} = 2^{11} - 1 = 2047$ *is not prime.*

Lucas-Lehmer Test

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Proof

i	$S_i \bmod 2047$

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1	4

Theorem

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Proof

i	$S_i \bmod 2047$
1	4
2	$(4^2 - 2) = 14 \bmod 2047 = 14$

Theorem

$M_{11} = 2^{11} - 1 = 2047$ *is not prime.*

Proof

i	$S_i \bmod 2047$
1	4
2	$(4^2 - 2) = 14 \bmod 2047 = 14$
3	$(14^2 - 2) = 194 \bmod 2047 = 194$

Lucas-Lehmer Test

Theorem

$M_{11} = 2^{11} - 1 = 2047$ is not prime.

Proof

i	$S_i \bmod 2047$
1	4
2	$(4^2 - 2) = 14 \bmod 2047 = 14$
3	$(14^2 - 2) = 194 \bmod 2047 = 194$
4	$(194^2 - 2) = 37634 \bmod 2047 = 788$

Theorem

$M_{11} = 2^{11} - 1 = 2047$ *is not prime.*

Proof

i	$S_i \bmod 2047$
1	4
2	$(4^2 - 2) = 14 \bmod 2047 = 14$
3	$(14^2 - 2) = 194 \bmod 2047 = 194$
4	$(194^2 - 2) = 37634 \bmod 2047 = 788$
5	$(788^2 - 2) = 620942 \bmod 2047 = 701$

$2^{11} - 1$ is not prime

Proof cont.

i	$S_i \bmod 2047$

$2^{11} - 1$ is not prime

Proof cont.

i	$S_i \bmod 2047$
6	$(701^2 - 2) = 491399 \bmod 2047 = 119$

$2^{11} - 1$ is not prime

Proof cont.

i	$S_i \bmod 2047$
6	$(701^2 - 2) = 491399 \bmod 2047 = 119$
7	$(119^2 - 2) = 14159 \bmod 2047 = 1877$

$2^{11}-1$ is not prime

Proof cont.

i	$S_i \bmod 2047$
6	$(701^2-2) = 491399 \bmod 2047 = 119$
7	$(119^2-2) = 14159 \bmod 2047 = 1877$
8	$(1877^2-2) = 3523127 \bmod 2047 = 240$

$2^{11} - 1$ is not prime

Proof cont.

i	$S_i \bmod 2047$
6	$(701^2 - 2) = 491399 \bmod 2047 = 119$
7	$(119^2 - 2) = 14159 \bmod 2047 = 1877$
8	$(1877^2 - 2) = 3523127 \bmod 2047 = 240$
9	$(240^2 - 2) = 57598 \bmod 2047 = 282$

$2^{11}-1$ is not prime

Proof cont.

i	$S_i \bmod 2047$
6	$(701^2 - 2) = 491399 \bmod 2047 = 119$
7	$(119^2 - 2) = 14159 \bmod 2047 = 1877$
8	$(1877^2 - 2) = 3523127 \bmod 2047 = 240$
9	$(240^2 - 2) = 57598 \bmod 2047 = 282$
10	$(282^2 - 2) = 79522 \bmod 2047 = 1736$

Theorem

$M_{31} = 2^{31} - 1 = 2147483647$ *is prime.*

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1	4

Theorem

$M_{31} = 2^{31} - 1 = 2147483647$ *is prime.*

Proof.

i	$S_i \bmod 2^{31} - 1$
1	4
2	14

Lucas-Lehmer Test

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$M_{31} = 2^{31} - 1 = 2147483647$ *is prime.*

Proof.

i	$S_i \bmod 2^{31} - 1$
1	4
2	14
3	194

Theorem

$M_{31} = 2^{31} - 1 = 2147483647$ *is prime.*

Proof.

i	$S_i \bmod 2^{31} - 1$
1	4
2	14
3	194
4	37634

Theorem

$M_{31} = 2^{31} - 1 = 2147483647$ *is prime.*

Proof.

i	$S_i \bmod 2^{31} - 1$
1	4
2	14
3	194
4	37634
5	1416317954

Theorem

$M_{31} = 2^{31} - 1 = 2147483647$ is prime.

Proof.

i	$S_i \bmod 2^{31} - 1$
1	4
2	14
3	194
4	37634
5	1416317954
6	669670838

Theorem

$M_{31} = 2^{31} - 1 = 2147483647$ *is prime.*

Proof.

i	$S_i \bmod 2^{31} - 1$
1	4
2	14
3	194
4	37634
5	1416317954
6	669670838
7	1937259419

Theorem

$M_{31} = 2^{31} - 1 = 2147483647$ *is prime.*

Proof.

i	$S_i \bmod 2^{31} - 1$
1	4
2	14
3	194
4	37634
5	1416317954
6	669670838
7	1937259419
8	425413602

$2^{31} - 1$ is prime

i	$S_i \bmod 2^{31} - 1$
9	842014276
10	12692426
11	2044502122
12	1119438707
13	1190075270
14	1450757861
15	877666528
16	630853853
17	940321271
18	512995887
19	692931217

$2^{31} - 1$ is prime

i	$S_i \bmod 2^{31} - 1$
20	1883625615
21	1992425718
22	721929267
23	27220594
24	1570086542
25	1676390412

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27	211987665

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28	1181536708
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23	27220594
24	1570086542
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26	1159251674
27	211987665
28	1181536708
29	65536
30	0

Theorem

If M_p is prime, then p is prime.

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Proof

By contradiction. Suppose p is composite. Then $p = ab$ for some $a, b > 1$. But then

$$\begin{aligned} 2^p - 1 = 2^{ab} - 1 &= (2^a)^b - 1 \\ &= (2^a - 1) \cdot (2^{a(b-1)} + 2^{a(b-2)} + \cdots + 2^a + 1). \end{aligned}$$

Since the last two factors are both greater than 1, $2^p - 1$ is composite, a contradiction.

Perfect Numbers

- A **perfect number** is a positive integer that is equal to the sum of its proper positive divisors, that is the sum of its positive divisors excluding the number itself.

Perfect Numbers

- A **perfect number** is a positive integer that is equal to the sum of its proper positive divisors, that is the sum of its positive divisors excluding the number itself.
- First Eight Perfect Numbers:

$$\begin{aligned} 6 &= 1+2+3 \\ 28 &= 1+2+4+7+14 \\ 496 &= 1+2+4+8+16+31+62+124+248 \end{aligned}$$

$$8128,\ 33550336,\ 8589869056,$$
$$137438691328,\ 2305843008139952128$$

Perfect Number Theorem

Theorem

An even positive integer n is perfect if and only if there exists a positive integer p such that $2^p - 1$ is prime and $n = 2^{p-1} \cdot (2^p - 1)$.

Proof

($\Leftarrow$) Let $n = 2^{p-1} \cdot (2^p - 1)$, where $2^p - 1$ is prime. Since 2^{p-1} and $2^p - 1$ are relatively prime, the sum of the divisors of n is equal to the sum of the divisors of 2^{p-1} times the sum of the divisors of $2^p - 1$. But the sum of the divisors of 2^{p-1} is

$$1 + 2 + \cdots + 2^{p-2} + 2^{p-1} = 2^p - 1$$

and the sum of the divisors of $2^p - 1$ is 2^p, since $2^p - 1$ is prime. And the product is

$$(2^p - 1) \cdot 2^p = 2 \cdot 2^{p-1}(2^p - 1) = 2n.$$

So the sum of the proper divisors of n is n and n is perfect.

Proof

($\Rightarrow$) The proof is left to the reader.

Email Address and Talk URL

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cooper@ucmo.edu

Talk:
http://www.math-cs.ucmo.edu/∼curtisc/talks/gimpskme/mersennekme.pdf