Mersenne Primes, GIMPS, and the LL Test

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Prime Numbers

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- Prime Numbers Less Than 100:

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ 41,$$
$$43,\ 47,\ 53,\ 59,\ 61,\ 67,\ 71,\ 73,\ 79,\ 83,\ 89,\ 97$$

Mersenne Numbers

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- Examples of Mersenne numbers are:

$$M2 = 2^2 - 1 = 3$$
$$M3 = 2^3 - 1 = 7$$
$$M5 = 2^5 - 1 = 31$$
$$M7 = 2^7 - 1 = 127$$
$$M11 = 2^{11} - 1 = 2047$$

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- $2047 = 2^{11} - 1 = 23 \times 89$.

1. Mersenne Primes
   - Primes
   - Mersenne Primes
2. History
   - Marin Mersenne
   - Edouard Lucas
   - Computer Era
3. 51st MP
   - M82589933
   - News on 51th Mersenne Prime
4. GIMPS
   - GIMPS
   - GIMPS People
   - GIMPS Links
5. LL Test
   - Lucas-Lehmer Test

Marin Mersenne

- Mersenne primes are named after a 17th-century French monk and mathematician

Marin Mersenne (1588-1648)

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- 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257

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- Mersenne compiled what was supposed to be a list of Mersenne primes with exponents up to 257.
- 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257
- His list was largely incorrect, as Mersenne mistakenly included M67 and M257 (which are composite), and omitted M61, M89, and M107 (which are prime).

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- Without finding a factor, Lucas demonstrated that M67 is actually composite.

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- $$\begin{array}{r} 193,707,721 \\ \times 761,838,257,287 \\ 147,573,952,589,676,412,927. \end{array}$$

List of Mersenne Primes

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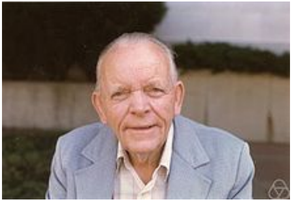
- Later Landon Curt Noll found M23209 (1979).

Noll and Lehmer

Landon Curt Noll



Derrick Henry Lehmer

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- He claimed the 100,000 dollar prize, awarded by the Electronic Frontier Foundation, for the first known prime with at least 10 million decimal digits.
- The prime was found on a Dell OptiPlex 745. This is the eighth Mersenne prime discovered at UCLA.

List of Known Mersenne Primes

- List of 51 Known Mersenne Primes - https://en.wikipedia.org/wiki/Mersenne_prime

M82589933

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- After less than 4 months and on just his fourth try, he discovered the new prime number.

News About 51th Mersenne Prime

- Official Press Release
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- John D. Cook
  https://www.johndcook.com/blog/2018/12/22/51st-mersenne-prime/

Digits of M82589933 by Landon Curt Noll

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UCM's Four Mersenne Primes

- M30402457
  https://www.mersenne.org/primes/?press=M30402457

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More About 49th Mersenne Prime

- Standupmaths
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- Jimmy Fallon
  https://www.facebook.com/kshbtv/videos/10153315475526190

Mersenne Buttons

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  cs.ucmo.edu/∼cnc8851/images/9.jpg

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- M74207281 Button
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Jumping GIFS

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- UCM GIF
  http://cs.ucmo.edu/∼cnc8851/images/14.gif

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- Scott Kurowski wrote the PrimeNet server that supports GIMPS. In 1997 he founded Entropia, a distributed computing software company.

GIMPS Statistics

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- UCM has over 700 computers performing LL-tests on Mersenne numbers.

Woltman, Kurowski, and Crandall

Woltman

Kurowski

Crandall

GIMPS Links

- The GIMPS home page can be found at: https://www.mersenne.org

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- A Mersenne Prime discussion forum can be found at: http://www.mersenneforum.org

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Let $S_1 = 4$ and

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Lucas-Lehmer Test

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Definition

Let $S_1 = 4$ and

$$S_{n+1} = S_n^2 - 2 \text{ for } n \geq 1.$$

- The first few terms of the S sequence are:

$$4,\ 14,\ 194,\ 37634,\ 1416317954,\ 2005956546822746114,\ 4023861667741036022825635656102100994, \ldots$$

Lucas-Lehmer Test

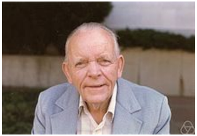
Lucas-Lehmer Test

Let p be a prime number. Then

$$M_p = 2^p - 1 \text{ is prime}$$
$$\text{if and only if}$$
$$S_{p-1} \bmod M_p = 0.$$

Lucas and Lehmer

Lucas



Lehmer

$2^{11} - 1$ is not prime

Theorem

$M_{11} = 2^{11} - 1 = 2047$ *is not prime.*

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i	$S_i \bmod 2047$

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Proof

i	$S_i \bmod 2047$
1	4

$2^{11} - 1$ is not prime

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i	$S_i \bmod 2047$
1	4
2	$(4^2 - 2) = 14 \bmod 2047 = 14$

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i	$S_i \bmod 2047$
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2	$(4^2 - 2) = 14 \bmod 2047 = 14$
3	$(14^2 - 2) = 194 \bmod 2047 = 194$

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i	$S_i \bmod 2047$
1	4
2	$(4^2 - 2) = 14 \bmod 2047 = 14$
3	$(14^2 - 2) = 194 \bmod 2047 = 194$
4	$(194^2 - 2) = 37634 \bmod 2047 = 788$

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i	$S_i \bmod 2047$
1	4
2	$(4^2 - 2) = 14 \bmod 2047 = 14$
3	$(14^2 - 2) = 194 \bmod 2047 = 194$
4	$(194^2 - 2) = 37634 \bmod 2047 = 788$
5	$(788^2 - 2) = 620942 \bmod 2047 = 701$

$2^{11} - 1$ is not prime

Proof cont.

i	$S_i \bmod 2047$

$2^{11} - 1$ is not prime

Proof cont.

i	S_i mod 2047
6	$(701^2 - 2) = 491399 \bmod 2047 = 119$

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Proof cont.

i	$S_i \bmod 2047$
6	$(701^2 - 2) = 491399 \bmod 2047 = 119$
7	$(119^2 - 2) = 14159 \bmod 2047 = 1877$

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Proof cont.

i	$S_i \bmod 2047$
6	$(701^2 - 2) = 491399 \bmod 2047 = 119$
7	$(119^2 - 2) = 14159 \bmod 2047 = 1877$
8	$(1877^2 - 2) = 3523127 \bmod 2047 = 240$

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Proof cont.

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6	$(701^2 - 2) = 491399 \bmod 2047 = 119$
7	$(119^2 - 2) = 14159 \bmod 2047 = 1877$
8	$(1877^2 - 2) = 3523127 \bmod 2047 = 240$
9	$(240^2 - 2) = 57598 \bmod 2047 = 282$

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Proof cont.

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7	$(119^2 - 2) = 14159 \bmod 2047 = 1877$
8	$(1877^2 - 2) = 3523127 \bmod 2047 = 240$
9	$(240^2 - 2) = 57598 \bmod 2047 = 282$
10	$(282^2 - 2) = 79522 \bmod 2047 = 1736$

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3	194
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1	4
2	14
3	194
4	37634
5	1416317954
6	669670838
7	1937259419
8	425413602

$2^{31} - 1$ is prime

i	$S_i \bmod (2^{31} - 1)$
9	842014276
10	12692426
11	2044502122
12	1119438707
13	1190075270
14	1450757861
15	877666528
16	630853853
17	940321271
18	512995887
19	692931217

$2^{31} - 1$ is prime

i	$S_i \bmod (2^{31} - 1)$
20	1883625615
21	1992425718
22	721929267
23	27220594
24	1570086542
25	1676390412

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26	1159251674
27	211987665
28	1181536708
29	65536
30	0

Sign of Penultimate Term

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- The OEIS sequence A123271 shows $\varepsilon(4, p)$ for each Mersenne prime M_p.

Sign of $\varepsilon(4, p)$

p	$\varepsilon(4,p)$	p	$\varepsilon(4,p)$	p	$\varepsilon(4,p)$
3	1	127	1	9941	1
5	1	521	-1	$11,213$	-1
7	-1	607	-1	$19,937$	1
13	1	1279	-1	$21,701$	-1
17	-1	2203	1	$23,209$	1
19	-1	2281	-1	$44,497$	-1
31	1	3217	-1	$86,243$	1
61	1	4253	1	$110,503$	1
89	-1	4423	-1	$132,049$	1
107	-1	9689	-1	$216,091$	-1

Sign of $\varepsilon(4, p)$ cont.

p	$\varepsilon(4, p)$
$25,964,951$	1
$30,402,457$	-1
$32,582,657$	-1
$37,156,667$	1
$42,643,801$	-1
$43,112,609$	1
$57,885,161$	-1
$74,207,281$	-1
$77,232,917$	1
$82,589,933$	-1

M127

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- Based on some theorems Lucas discovered and properties of Fibonacci numbers, his hand calculations boiled down to showing that if $r_1 = 3$, and

$$r_{k+1} = r_k^2 - 2 \text{ for } k \geq 1,$$

then if

$$r_{126} \bmod M127 = 0,$$

then M127 is prime.

M127

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- To see how Lucas did this, we will reduce the problem.
- We will show that $M7 = 2^7 - 1 = 127$ is prime.
- For our reduced problem, we will play Lucas' game on a 7×7 chessboard.

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- $r_6 = (16^2 - 2) \bmod 127 = 0.$
- Therefore, M7 is prime.

Chessboard

- The 7×7 chessboard will store the calculations in base 2 (modulo 127). Columns on the board will represent powers of 2 and the rows will store the product of a single base 2 digit in r_k times the base 2 number r_k. Lucas used a pawn or no pawn to represent a 1 or 0 on the board, respectively.

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- Initially, the top row will contain $r_1 = 3$.

Chessboard

- If the top row contained r_k, Lucas would square r_k with the following moves.

Chessboard

- If the top row contained r_k, Lucas would square r_k with the following moves.
- He would do standard multiplication to populate the board with pawns. Each row corresponds to putting a shift of the top row in the row or having no pawns in the row, depending on whether there is a pawn in the corresponding column of the top row or not. Because Lucas is doing the calculations modulo 127, the columns wrap around the chessboard.

Pawn Moves

- He would then subtract 2 (once), usually by taking a pawn away from Column f. In the game, two pawns in the same column would be equivalent to removing those two pawns and replacing them by one pawn in the next column to the left. The column to the left of the left-most column is the right-most column.

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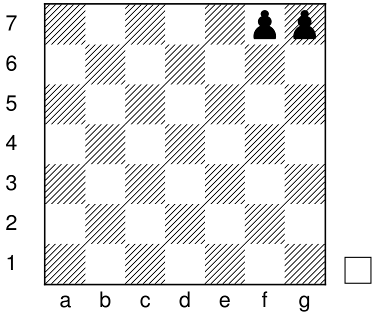
- Lucas kept this game going until he didn't have two pawns in any column. Then he would slide each pawn in a column to the top row. This would be his r_{k+1}.

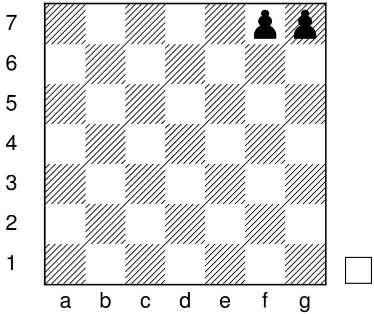
Lucas Game

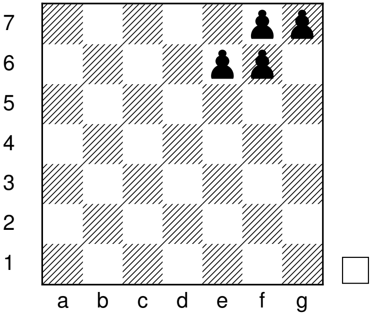
Lucas started the game with $r_1 = 3$.
On the chessboard, that would be:



Lucas Game

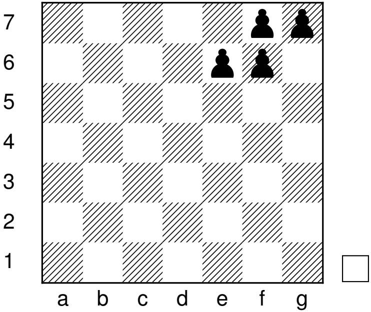
Squaring $r_1 = 3$ would result in the following chessboard.

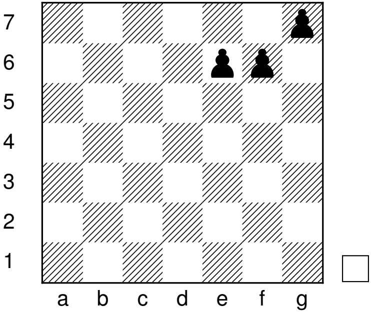




Lucas Game

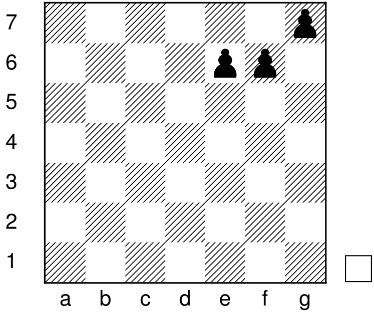
We can subtract 2 by removing a pawn from Column f. That would result in the following chessboard.

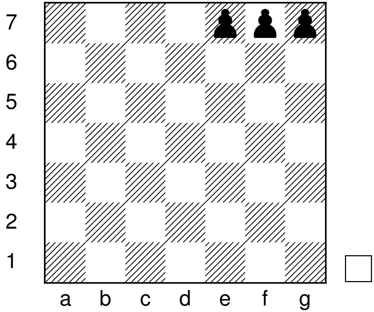




Lucas Game

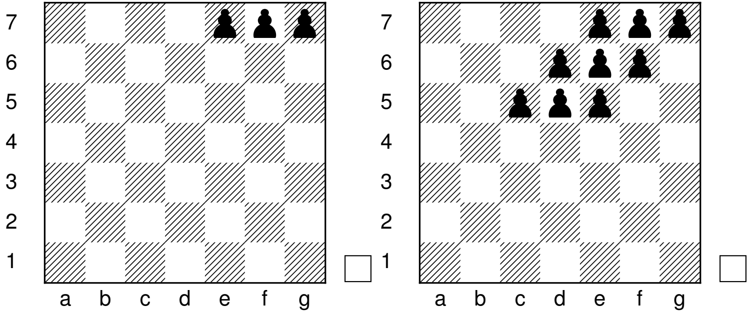
Pushing all the pawns to the top row would result in the following chessboard which is $r_2 = 7$.





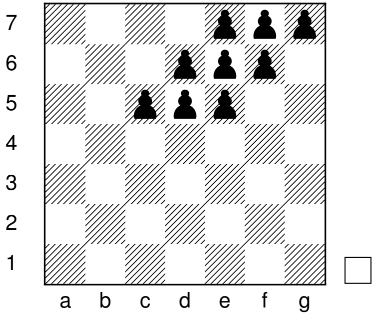
Lucas Game

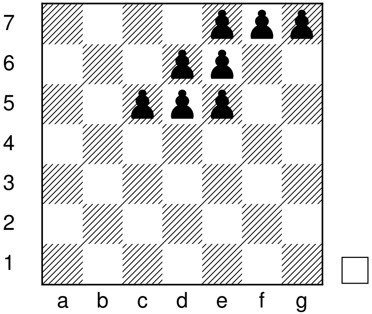
Now we need to square $r_2 = 7$. This would result in the following chessboard.



Lucas Game

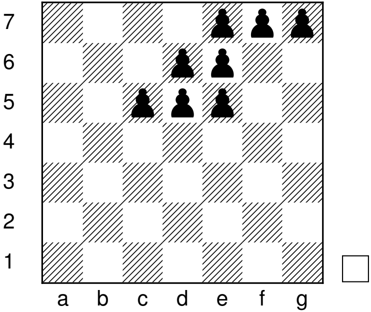
Subtracting 2 would result in the following chessboard.

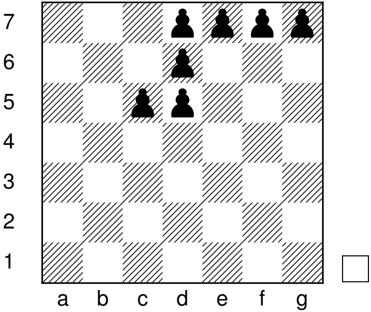




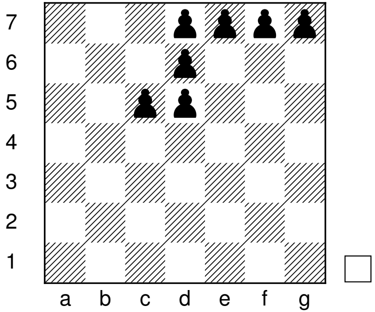
Lucas Game

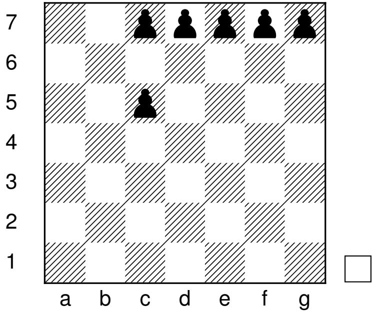
We now do the game moves where we replace two pawns in a column by one pawn in the column to the left. Here are the steps in the game.



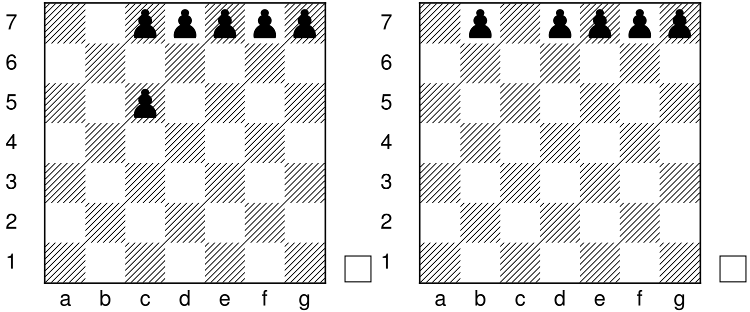


Lucas Game



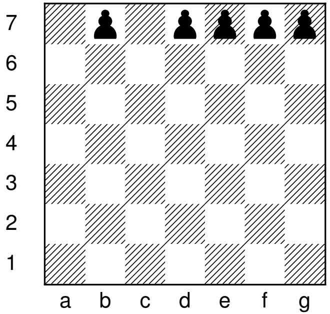


Lucas Game



Lucas Game

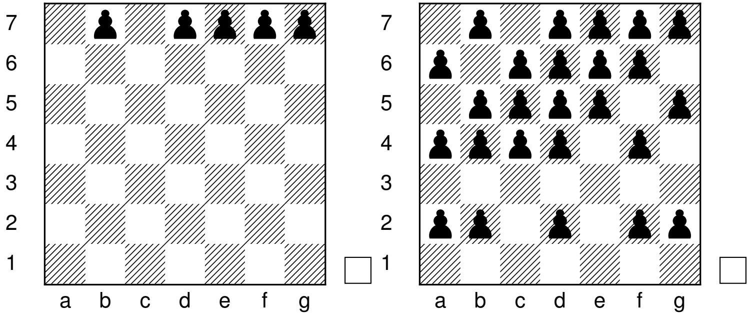
The final chessboard with $r_3 = 47$ would be the following.



$\square$

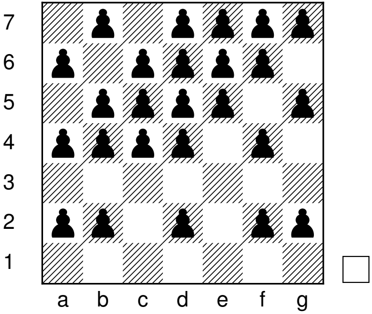
Lucas Game

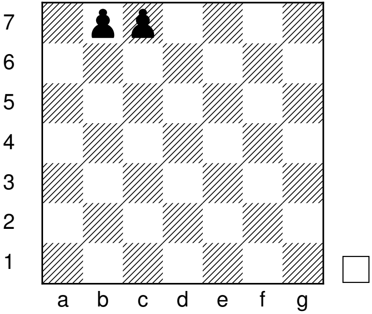
Squaring $r_3 = 47$, we obtain the following chessboard.



Lucas Game

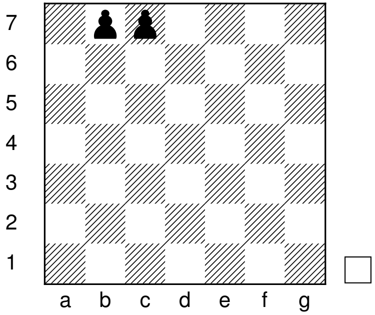
Subtracting 2 and reducing, we obtain the following chessboard.

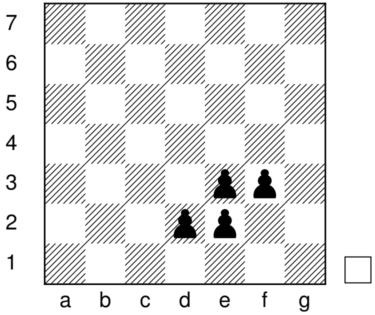




Lucas Game

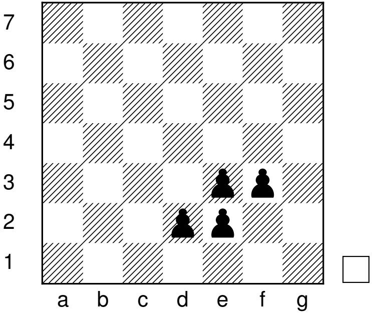
Squaring $r_4 = 48$, we obtain the following chessboard.

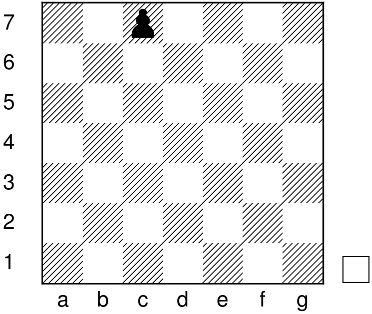




Lucas Game

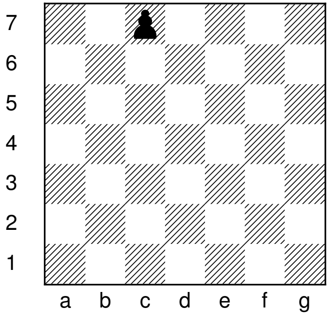
Subtracting 2 and reducing, we obtain the following chessboard.

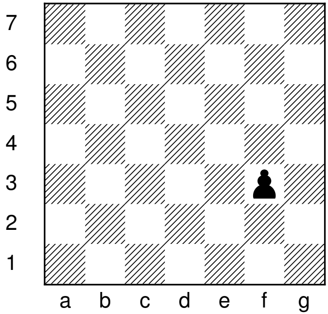




Lucas Game

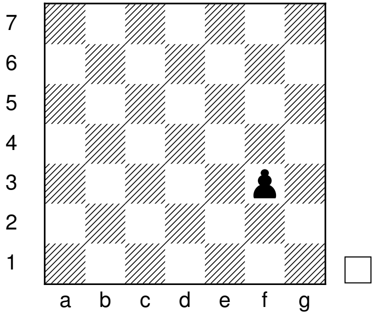
Squaring $r_5 = 16$, we obtain the following chessboard.

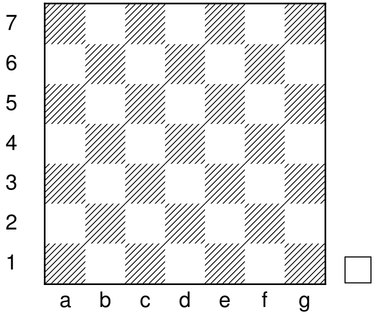




Lucas Game

Subtracting 2 from the result gives $r_6 = 0$.





Lucas Game

Thus, $r_6 = 0$.

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Therefore, $M7 = 2^7 - 1 = 127$ is a Mersenne prime.

5 Fun Facts

5 Fun Facts on GIMPS

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- 1. The largest known prime as of June 11, 2019 is:

$$2^{82,589,933} - 1.$$

It was discovered by Patrick LaRoche, George Woltman, Aaron Blosser, et al. (GIMPS) on December 7, 2018 and has 24,862,048 decimal digits.

2nd Fact

- 2. The Great Internet Mersenne Prime Search (GIMPS) is a volunteer organization devoted to the search for large Mersenne primes. George Woltman founded GIMPS in 1996 and created the software used to search for large Mersenne primes. The group has found 17 world-record prime numbers over its 23 years of existence. The software can be freely downloaded at:

www.mersenne.org

3rd Fact

- 3. Marin Mersenne and Eduoard Lucas are mathematicians who researched Mersenne primes. Mersenne was a 17th century French monk. In 1876, Lucas discovered and proved that $2^{127} - 1$ is prime. This prime is the largest prime proved without the use of a computer. His method of proof, using the Lucas-Lehmer Test, is essentially the technique used today to prove Mersenne numbers are prime.

4th Fact

- 4. The University of Central Missouri has found 4 Mersenne primes as a participant in GIMPS. They are:

$$2^{30,402,457} - 1,$$
$$2^{32,582,657} - 1,$$
$$2^{57,885,161} - 1,$$
$$2^{74,207,281} - 1.$$

They were found in 2005, 2006, 2013, and 2016, respectively. At the time, each of them was the largest known prime number.

5th Fact

- 5. The Electronic Frontier Foundation (EFF) has offered a $150,000 prize for the discovery of the first one-hundred million digit prime number. EFF's motivation is to encourage research in computational number theory related to large primes.

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- Talk:
  cs.ucmo.edu/∼cnc8851/talks/gimpsmsa5/gimpsmsa5.pdf