Some High Degree Generalized Fibonacci Identities

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Outline

Let $\{F_n\}$ and $\{L_n\}$ be the Fibonacci and Lucas sequences, respectively.

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Gelin stated and Cesáro proved that for integers $n \geq 2$,

$$F_{n-2}F_{n-1}F_{n+1}F_{n+2} - F_n^4 = -1.$$

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Gelin stated and Cesáro proved that for integers $n \geq 2$,

$$F_{n-2}F_{n-1}F_{n+1}F_{n+2} - F_n^4 = -1.$$

To generalize this identity, we need the following definition due to Horadam.

Definition

Let $\{W_n\}$ be defined by $W_0 = a$, $W_1 = b$, and $W_n = pW_{n-1} - qW_{n-2}$ for $n \geq 2$, where a, b, p, and q are integers and $q \neq 0$. Let $e = pab - qa^2 - b^2$.

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Melham and Shannon generalized the Gelin-Cesáro identity by proving that for integers $n \geq 2$,

$$W_{n-2}W_{n-1}W_{n+1}W_{n+2} - W_n^4 = eq^{n-2}(p^2+q)W_n^2 + e^2q^{2n-3}p^2. \tag{1}$$

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In this paper, we will generalize and prove some similar high degree generalized Fibonacci identities.

Outline

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Definition

Let $\{U_n\}$ be defined by $U_0 = 0$, $U_1 = 1$, and $U_n = pU_{n-1} - qU_{n-2}$ for $n \geq 2$, where p and q are integers and $q \neq 0$.

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Let $\{U_n\}$ be defined by $U_0 = 0$, $U_1 = 1$, and $U_n = pU_{n-1} - qU_{n-2}$ for $n \geq 2$, where p and q are integers and $q \neq 0$.

The sequence $\{U_n\}$ is the fundamental sequence of Lucas. With this definition, we can state a generalization of the Melham and Shannon identity.

Theorem

Let r and s be positive integers and $n \geq r+s$ be an integer. Then

$$\begin{aligned} & W_{n-r-s}W_{n-r}W_{n+r}W_{n+r+s} \\ & = W_n^4 + eq^{n-r-s}(q^sU_r^2 + U_{r+s}^2)W_n^2 + e^2q^{2n-2r-s}U_r^2U_{r+s}^2. \end{aligned} \quad (2)$$

Theorem

Let r and s be positive integers and $n \geq r + s$ be an integer. Then

$$W_{n-r-s}W_{n-r}W_{n+r}W_{n+r+s} = W_n^4 + eq^{n-r-s}(q^s U_r^2 + U_{r+s}^2)W_n^2 + e^2 q^{2n-2r-s} U_r^2 U_{r+s}^2. \quad (2)$$

We note that when $r = 1$ and $s = 1$, this is the Melham and Shannon identity.

The proof of (2) is similar to the proof of the Melham and Shannon identity. Before we begin the proof (2), we require more definitions and a lemma from Melham and Shannon.

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Definition

Let $\{Y_n\}$ be defined by $Y_0 = a_1$, $Y_1 = b_1$, and $Y_n = pY_{n-1} - qY_{n-2}$ for $n \geq 2$, where a_1, b_1, p, and q are integers and $q \neq 0$.

Definition

Let s be a nonnegative integer. Let

$$\Psi(s) = (pa_1b - qaa_1 - bb_1)U_s + (ab_1 - a_1b)U_{s+1}.$$

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Lemma

Let n be a nonnegative integer and r and s be positive integers. Then

$$W_n Y_{n+r+s} - W_{n+r} Y_{n+s} = \Psi(s) q^n U_r. \tag{3}$$

In (3), replacing n by $n - r$ and s by r gives

$$W_{n-r}Y_{n+r} - W_nY_n = \Psi(r)q^{n-r}U_r. \tag{4}$$

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$$W_{n-r}Y_{n+r} - W_nY_n = \Psi(r)q^{n-r}U_r. \tag{4}$$

Replacing r by $r+s$ in (4), we have

$$W_{n-r-s}Y_{n+r+s} - W_nY_n = \Psi(r+s)q^{n-r-s}U_{r+s}. \tag{5}$$

In (3), replacing n by $n-r$ and s by r gives

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$$W_{n-r-s}Y_{n+r+s} - W_nY_n = \Psi(r+s)q^{n-r-s}U_{r+s}. \tag{5}$$

Adding (4) and (5) gives

$$\begin{aligned} &W_{n-r}Y_{n+r} + W_{n-r-s}Y_{n+r+s} \\ &= 2W_nY_n + \Psi(r)q^{n-r}U_r + \Psi(r+s)q^{n-r-s}U_{r+s}. \end{aligned} \tag{6}$$

Subtracting (5) from (4) gives

$$W_{n-r}Y_{n+r} - W_{n-r-s}Y_{n+r+s} = \Psi(r)q^{n-r}U_r - \Psi(r+s)q^{n-r-s}U_{r+s}. \tag{7}$$

Subtracting (5) from (4) gives

$$W_{n-r}Y_{n+r} - W_{n-r-s}Y_{n+r+s} = \Psi(r)q^{n-r}U_r - \Psi(r+s)q^{n-r-s}U_{r+s}. \tag{7}$$

Squaring (6) and subtracting the square of (7), we obtain

$$\begin{aligned} &4W_{n-r-s}W_{n-r}Y_{n+r}Y_{n+r+s} \\ &= 4W_n^2Y_n^2 + 4q^{n-r-s}(q^s\Psi(r)U_r + \Psi(r+s)U_{r+s})W_nY_n \\ &+ 4\Psi(r)\Psi(r+s)q^{2n-2r-s}U_rU_{r+s}. \end{aligned} \tag{8}$$

Divide both sides of the equation by 4. Now, if $(a_1, b_1) = (a, b)$, then $\{W_n\} = \{Y_n\}$, $\Psi(r) = eU_r$, and $\Psi(r+s) = eU_{r+s}$. Substituting these quantities in (8), we see that (8) becomes (2). This is what we wanted to prove.

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Theorem

Let r and s be positive integers and $n \geq r+s$ be an integer. Then

$$W_{n-r-s}W_{n-r}W_{n+r}W_{n+r+s}$$
$$= W_n^4 + eq^{n-r-s}(q^s U_r^2 + U_{r+s}^2)W_n^2 + e^2 q^{2n-2r-s} U_r^2 U_{r+s}^2.$$

Note that if $r = 1$ and $s = 1$, then (2) becomes (1). Also, note that if $a = 0$, $b = 1$, $p = 1$, and $q = -1$, then $\{W_n\} = \{F_n\}$.

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Thus from (3), we have the following identity for Fibonacci numbers.

$$F_{n-r-s}F_{n-r}F_{n+r}F_{n+r+s}$$
$$= F_n^4 + (-1)^{n-r-s-1}((-1)^s F_r^2 + F_{r+s}^2)F_n^2 + (-1)^s F_r^2 F_{r+s}^2.$$

Outline

Theorem

Let r and s be positive integers and $n \geq r + s$ be an integer. Then

$$\begin{aligned} &3W_{n-r-s}W_{n-r}^2W_{n+r}^2W_{n+r+s} + W_{n-r-s}^3W_{n+r+s}^3 \\ &= 4W_n^6 + 6eq^{n-r-s}(q^sU_r^2 + U_{r+s}^2)W_n^4 \\ &+ 3e^2q^{2n-2r-2s}(q^{2s}U_r^4 + 2q^sU_r^2U_{r+s}^2 + U_{r+s}^4)W_n^2 \\ &+ e^3q^{3n-3r-3s}(3q^{2s}U_r^4U_{r+s}^2 + U_{r+s}^6). \end{aligned} \tag{9}$$

Again, the proof of this Theorem is similar to the proof of (1), but with a few modifications.

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We obtain

$$
\begin{aligned}
&6W_{n-r}^2 Y_{n+r}^2 W_{n-r-s} Y_{n+r+s} + 2W_{n-r-s}^3 Y_{n+r+s}^3 \qquad (10)\\
&= 8W_n^3 Y_n^3 + 12W_n^2 Y_n^2 \Psi(r) q^{n-r} U_r + 12W_n^2 Y_n^2 \Psi(r+s) q^{n-r-s} U_{r+s}\\
&+ 6W_n Y_n \Psi(r)^2 q^{2n-2r} U_r^2 + 12W_n Y_n \Psi(r) q^{2n-2r-s} \Psi(r+s) U_r U_{r+s}\\
&+ 6W_n Y_n \Psi(r+s)^2 q^{2n-2r-2s} U_{r+s}^2 + 6q^{3n-3r-s} \Psi(r)^2 \Psi(r+s) U_r^2 U_{r+s}\\
&+ 2q^{3n-3r-3s} \Psi(r+s)^3 U_{r+s}^3.
\end{aligned}
$$

Divide both sides of the equation by 2. Again, if $(a_1, b_1) = (a, b)$, then $\{W_n\} = \{Y_n\}$, $\Psi(r) = eU_r$, and $\Psi(r+s) = eU_{r+s}$.

Divide both sides of the equation by 2. Again, if $(a_1, b_1) = (a, b)$, then $\{W_n\} = \{Y_n\}$, $\Psi(r) = eU_r$, and $\Psi(r+s) = eU_{r+s}$.

Substituting these quantities in (10), we see that (10) becomes (9). This is what we wanted to prove.

Divide both sides of the equation by 2. Again, if $(a_1, b_1) = (a, b)$, then $\{W_n\} = \{Y_n\}$, $\Psi(r) = eU_r$, and $\Psi(r+s) = eU_{r+s}$.

Substituting these quantities in (10), we see that (10) becomes (9). This is what we wanted to prove.

Theorem

Let r and s be positive integers and $n \geq r+s$ be an integer. Then

$$\begin{aligned}
&3W_{n-r-s}W_{n-r}^2W_{n+r}^2W_{n+r+s} + W_{n-r-s}^3W_{n+r+s}^3 \\
&= 4W_n^6 + 6eq^{n-r-s}(q^sU_r^2 + U_{r+s}^2)W_n^4 \\
&+ 3e^2q^{2n-2r-2s}(q^{2s}U_r^4 + 2q^sU_r^2U_{r+s}^2 + U_{r+s}^4)W_n^2 \\
&+ e^3q^{3n-3r-3s}(3q^{2s}U_r^4U_{r+s}^2 + U_{r+s}^6).
\end{aligned}$$

Again, if $a = 0$, $b = 1$, $p = 1$, and $q = -1$, then $\{W_n\} = \{F_n\}$.

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Thus from (9), we have the following identity for Fibonacci numbers.

$$\begin{aligned}
&3F_{n-r-s}F_{n-r}^2F_{n+r}^2F_{n+r+s} + F_{n-r-s}^3F_{n+r+s}^3 \\
&= 4F_n^6 + 6(-1)^{n-r-s-1}((-1)^sF_r^2 + F_{r+s}^2)F_n^4 \\
&+ 3(F_r^4 + 2(-1)^sF_r^2F_{r+s}^2 + F_{r+s}^4)F_n^2 \\
&+ (-1)^{n-r-s-1}(3F_r^4F_{r+s}^2 + F_{r+s}^6).
\end{aligned}$$

Outline

Theorem

Let r and s be positive integers, $k \geq 2$ be an integer, and $n \geq r + s$ be an integer. Then

$$2\sum_{i \geq 1} \binom{k}{2i-1} (W_{n-r}W_{n+r})^{k+1-2i} (W_{n-r-s}W_{n+r+s})^{2i-1} \tag{11}$$

$$= \sum_{i=0}^{k-1} \binom{k}{i} (2W_n^2)^{k-i} e^i q^{in-ir-is} (q^s U_r^2 + U_{r+s}^2)^i$$

$$+ 2e^k q^{kn-kr-ks} \sum_{i \geq 1} \binom{k}{2i-1} q^{(k+1-2i)s} U_r^{2(k+1-2i)} U_{r+s}^{2(2i-1)}.$$

Again, the proof of this Theorem is similar to the proof of (1), but with a few modifications.

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We start the proof of this Theorem as we started the proof of the other Theorem. But, this time, we consider (7) and (8) and let $(a_1, b_1) = (a, b)$. Then $\{W_n\} = \{Y_n\}$, $\Psi(r) = eU_r$, and $\Psi(r+s) = eU_{r+s}$. Substituting these quantities in (7) and (8), we obtain

$$W_{n-r}W_{n+r} + W_{n-r-s}W_{n+r+s} = 2W_n^2 + eq^{n-r}U_r^2 + eq^{n-r-s}U_{r+s}^2 \tag{12}$$

and

$$W_{n-r}W_{n+r} - W_{n-r-s}W_{n+r+s} = eq^{n-r}U_r^2 - eq^{n-r-s}U_{r+s}^2. \tag{13}$$

Instead of squaring, we raise (12) to the kth power and subtract (13) raised to the kth power and using polynomial expansion, we obtain

$$(W_{n-r}W_{n+r} + W_{n-r-s}W_{n+r+s})^k - (W_{n-r}W_{n+r} - W_{n-r-s}W_{n+r+s})^k$$

$$= \sum_{i=0}^{k-1} \binom{k}{i} (2W_n^2)^{k-i} (eq^{n-r}U_r^2 + eq^{n-r-s}U_{r+s}^2)^i$$

$$+ (eq^{n-r}U_r^2 + eq^{n-r-s}U_{r+s}^2)^k - (eq^{n-r}U_r^2 - eq^{n-r-s}U_{r+s}^2)^k.$$

Expanding the products on both sides of the equation and collecting and canceling terms gives

$$2\sum_{i\geq 1}\binom{k}{2i-1}(W_{n-r}W_{n+r})^{k+1-2i}(W_{n-r-s}W_{n+r+s})^{2i-1}$$

$$=\sum_{i=0}^{k-1}\binom{k}{i}(2W_n^2)^{k-i}e^i q^{in-ir-is}(q^s U_r^2+U_{r+s}^2)^i$$

$$+2e^k\sum_{i\geq 1}\binom{k}{2i-1}(q^{n-r}U_r^2)^{k+1-2i}(q^{n-r-s}U_{r+s}^2)^{2i-1}.$$

Simplifying some more, we obtain

$$2\sum_{i\geq 1}\binom{k}{2i-1}(W_{n-r}W_{n+r})^{k+1-2i}(W_{n-r-s}W_{n+r+s})^{2i-1}$$
$$=\sum_{i=0}^{k-1}\binom{k}{i}(2W_n^2)^{k-i}e^i q^{in-ir-is}(q^s U_r^2+U_{r+s}^2)^i$$
$$+2e^k q^{kn-kr-ks}\sum_{i\geq 1}\binom{k}{2i-1}q^{(k+1-2i)s}U_r^{2(k+1-2i)}U_{r+s}^{2(2i-1)}.$$

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This is (11) and what we wanted to prove.

Theorem

Let r and s be positive integers, $k \geq 2$ is an integer, and $n \geq r + s$ is an integer. Then

$$2\sum_{i\geq 1}\binom{k}{2i-1}(W_{n-r}W_{n+r})^{k+1-2i}(W_{n-r-s}W_{n+r+s})^{2i-1}$$

$$=\sum_{i=0}^{k-1}\binom{k}{i}(2W_n^2)^{k-i}e^i q^{in-ir-is}(q^s U_r^2 + U_{r+s}^2)^i$$

$$+2e^k q^{kn-kr-ks}\sum_{i\geq 1}\binom{k}{2i-1}q^{(k+1-2i)s}U_r^{2(k+1-2i)}U_{r+s}^{2(2i-1)}.$$

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Thus from (11), we have the following identity for Fibonacci numbers.

$$2\sum_{i\geq 1}\binom{k}{2i-1}(F_{n-r}F_{n+r})^{k+1-2i}(F_{n-r-s}F_{n+r+s})^{2i-1}$$
$$=\sum_{i=0}^{k-1}\binom{k}{i}(2F_n^2)^{k-i}(-1)^{in-ir-is+i}((-1)^sF_r^2+F_{r+s}^2)^i$$
$$+2(-1)^k(-1)^{kn-kr-ks}\sum_{i\geq 1}\binom{k}{2i-1}(-1)^{(k+1-2i)s}F_r^{2(k+1-2i)}F_{r+s}^{2(2i-1)}.$$

Outline

Next, we start with another fourth degree Fibonacci identity. If n is a nonnegative integer, then

$$F_n F_{n+4}^3 - F_{n+2}^3 F_{n+6} = (-1)^{n+1} F_{n+3} L_{n+3}. \tag{14}$$

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To state a generalization to (14), we need a definition due to Rabinowitz.

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To state a generalization to (14), we need a definition due to Rabinowitz.

Definition

Let n be an integer. Then

$$X_n = W_{n+1} - qW_{n-1}.$$

The sequence $\{X_n\}$ may be considered to be a companion sequence to $\{W_n\}$, in the same sense that the Lucas sequence is the companion of the Fibonacci sequence.

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Theorem

Let n be a nonnegative integer. Then

$$W_n W_{n+4}^3 - W_{n+2}^3 W_{n+6} = ep^3 q^n W_{n+3} X_{n+3}. \tag{15}$$

Let n be a nonnegative integer. Let $x = W_n$ and $y = W_{n+1}$.

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Then, after some substitutions and collecting terms, we have

$$
\begin{aligned}
W_n &= x \\
W_{n+1} &= y \\
W_{n+2} &= py - qx \\
W_{n+3} &= (p^2 - q)y - pqx \\
W_{n+4} &= (p^3 - 2pq)y + (-p^2q + q^2)x \\
W_{n+5} &= (p^4 - 3p^2q + q^2)y + (-p^3q + 2pq^2)x \\
W_{n+6} &= (p^5 - 4p^3q + 3pq^2)y + (-p^4q + 3p^2q^2 - q^3)x \\
X_{n+3} &= (p^3 - 3pq)y + (-p^2q + 2q^2)x.
\end{aligned}
$$

We need one more quantity, eq^n. We have that

$$eq^n = W_n W_{n+2} - W_{n+1}^2 = x(py - qx) - y^2 = -qx^2 + pxy - y^2.$$

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After substitutions and some algebra, the left side of (15) simplifies to

$$\begin{aligned}
&(-p^6q^3 + 2p^4q^4)x^4 + (3p^7q^2 - 8p^5q^3 + 2p^3q^4)x^3y \\
&+ (-3p^8q + 9p^6q^2 - 3p^4q^3)x^2y^2 \\
&+ (p^9 - 2p^7q - 3p^5q^2 + 2p^3q^3)xy^3 \\
&+ (-p^8 + 4p^6q - 3p^4q^2)y^4.
\end{aligned}$$

Again, after substitutions and some algebra, the right side of (15) simplifies to

$$
\begin{aligned}
&(-p^6q^3 + 2p^4q^4)x^4 + (3p^7q^2 - 8p^5q^3 + 2p^3q^4)x^3y \\
&+ (-3p^8q + 9p^6q^2 - 3p^4q^3)x^2y^2 \\
&+ (p^9 - 2p^7q - 3p^5q^2 + 2p^3q^3)xy^3 \\
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&+ (-3p^8q + 9p^6q^2 - 3p^4q^3)x^2y^2 \\
&+ (p^9 - 2p^7q - 3p^5q^2 + 2p^3q^3)xy^3 \\
&+ (-p^8 + 4p^6q - 3p^4q^2)y^4.
\end{aligned}$$

Therefore, the left side and right side of (15) are equal. This completes the proof of the theorem.

Outline

Finally, we present a fifth degree Fibonacci identity. If n is a nonnegative integer, then

$$F_n^2 F_{n+5}^3 - F_{n+1}^3 F_{n+6}^2 = (-1)^{n+1} L_{n+3}^3. \tag{16}$$

A generalization of (16) is presented in the following theorem.

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Theorem

Let n be a nonnegative integer. Then

$$W_n^2 W_{n+5}^3 - W_{n+1}^3 W_{n+6}^2 \tag{17}$$
$$= eq^n X_{n+3}((2p^3 - 3pq)W_{n+3}^2 + (p^7 - 2p^5 q + p^3 q^2)eq^n).$$

Again, we require the following quantities.

$$
\begin{aligned}
W_n &= x \\
W_{n+1} &= y \\
W_{n+2} &= py - qx \\
W_{n+3} &= (p^2 - q)y - pqx \\
W_{n+4} &= (p^3 - 2pq)y + (-p^2q + q^2)x \\
W_{n+5} &= (p^4 - 3p^2q + q^2)y + (-p^3q + 2pq^2)x \\
W_{n+6} &= (p^5 - 4p^3q + 3pq^2)y + (-p^4q + 3p^2q^2 - q^3)x \\
X_{n+3} &= (p^3 - 3pq)y + (-p^2q + 2q^2)x.
\end{aligned}
$$

After some substitutions and collecting terms, the left side of (17) simplifies to

$$\begin{aligned}
&(-p^9q^3 + 6p^7q^4 - 12p^5q^5 + 8p^3q^6)x^5 \\
&+ (3p^{10}q^2 - 21p^8q^3 + 51p^6q^4 - 48p^4q^5 + 12p^2q^6)x^4y \\
&+ (-3p^11q + 24p^9q^2 - 69p^7q^3 + 84p^5q^4 - 39p^3q^5 + 6pq^6)x^3y^2 \\
&+ (p^{12} - 9p^{10}q + 29p^8q^2 - 39p^6q^3 + 19p^4q^4 - 3p^2q^5)x^2y^3 \\
&+ (2p^9q - 14p^7q^2 + 32p^5q^3 - 26p^3q^4 + 6pq^5)xy^4 \\
&+ (-p^{10} + 8p^8q - 22p^6q^2 + 24p^4q^3 - 9p^2q^4)y^5.
\end{aligned}$$

Again, after substitutions and some algebra, the right side of (17) simplifies to

$$\begin{aligned}
&(-p^9q^3 + 6p^7q^4 - 12p^5q^5 + 8p^3q^6)x^5 \\
&+ (3p^{10}q^2 - 21p^8q^3 + 51p^6q^4 - 48p^4q^5 + 12p^2q^6)x^4y \\
&+ (-3p^{1}1q + 24p^9q^2 - 69p^7q^3 + 84p^5q^4 - 39p^3q^5 + 6pq^6)x^3y^2 \\
&+ (p^{12} - 9p^{10}q + 29p^8q^2 - 39p^6q^3 + 19p^4q^4 - 3p^2q^5)x^2y^3 \\
&+ (2p^9q - 14p^7q^2 + 32p^5q^3 - 26p^3q^4 + 6pq^5)xy^4 \\
&+ (-p^{10} + 8p^8q - 22p^6q^2 + 24p^4q^3 - 9p^2q^4)y^5.
\end{aligned}$$

Again, after substitutions and some algebra, the right side of (17) simplifies to

$$
\begin{aligned}
&(-p^9q^3 + 6p^7q^4 - 12p^5q^5 + 8p^3q^6)x^5 \\
&+ (3p^{10}q^2 - 21p^8q^3 + 51p^6q^4 - 48p^4q^5 + 12p^2q^6)x^4y \\
&+ (-3p^1 1q + 24p^9q^2 - 69p^7q^3 + 84p^5q^4 - 39p^3q^5 + 6pq^6)x^3y^2 \\
&+ (p^{12} - 9p^{10}q + 29p^8q^2 - 39p^6q^3 + 19p^4q^4 - 3p^2q^5)x^2y^3 \\
&+ (2p^9q - 14p^7q^2 + 32p^5q^3 - 26p^3q^4 + 6pq^5)xy^4 \\
&+ (-p^{10} + 8p^8q - 22p^6q^2 + 24p^4q^3 - 9p^2q^4)y^5.
\end{aligned}
$$

Therefore, the left side and right side of (17) are equal. This completes the proof of the theorem.

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