Niven Numbers and Sudoku Puzzles

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Introduction

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- The title of the talk was:
  On Using a Corollary to Chebyshev's Inequality for the Investigation of Natural Density

Introduction cont.

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- We did not give our talk. However, I did put the talk on my curriculum vita.
- In this talk, the Niven number work is joint with Bob Kennedy. The Sudoku work is joint with Hang Chen.

History of Niven Numbers

- In 1977, Robert Kennedy attended the 5th Annual Mathematics Conference at Miami University in Oxford, OH.

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- The conference theme was Number Theory and the Buckingham Scholar for the conference was Ivan Niven.

Ivan Niven

History of Niven Numbers cont.

- Professor Niven mentioned a question which appeared in the children's page of a newspaper. The question was:

  Find a whole number which is twice the sum of its digits.

History of Niven Numbers cont.

- Professor Niven mentioned a question which appeared in the children's page of a newspaper. The question was:

  Find a whole number which is twice the sum of its digits.
- The answer to his question is 18, since the sum of the decimal digits of 18 is 9 and 18 is 2 times 9.

History of Niven Numbers cont.

- My colleague, Bob Kennedy, was so inspired by Ivan Niven that he made the following definition in honor of Dr. Niven.

Definition (Niven Number)

A positive integer is a Niven number if it is divisible by the sum of its decimal digits.

History of Niven Numbers cont.

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- Hence, the birth of Niven numbers.

Examples of Niven Numbers

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$$1 - 10,\ 12,\ 20,\ 21,\ 1729,\ \text{and } 4050.$$

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- Some other examples of Niven numbers are:

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The Niven numbers less than or equal to 200 are:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 18, 20, 21, 24,
27, 30, 36, 40, 42, 45, 48, 50, 54, 60, 63, 70, 72,
80, 81, 84, 90, 100, 102, 108, 110, 111, 112, 114, 117,
120, 126, 132, 133, 135, 140, 144, 150, 152, 153, 156,
162, 171, 180, 190, 192, 195, 198, 200, . . .

Niven Number Facts

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- 1. Any number with digital sum 3 is a Niven number.

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- 1. Any number with digital sum 3 is a Niven number.
- 2. Any number with digital sum 9 is Niven.

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- 1. Any number with digital sum 3 is a Niven number.
- 2. Any number with digital sum 9 is Niven.
- 3. An even number with digital sum 2 is Niven.

Definition of $s(n)$

To study Niven numbers, we require the following definition.

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$$s(10) = 1, \;\; s(23) = 5, \;\; s(1729) = 19, \;\; \text{and} \;\; s(123456789) = 45.$$

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- Let us investigate whether or not $n!$ is Niven for any positive integer n.

Table of Factorials and Digital Sums

Here is a table of n, $n!$, and $s(n!)$

n	$n!$	$s(n!)$
1	1	1
2	2	2
3	6	6
4	24	6
5	120	3
6	720	9
7	5040	9
8	40320	9
9	362880	27
10	3628800	27

Table of Factorials and Digital Sums cont.

Here is a table of n, $n!$, and $s(n!)$

n	$n!$	$s(n!)$
11	39916800	36
12	479001600	27
13	6227020800	27
14	87178291200	45
15	1307674368000	45
16	20922789888000	63
17	355687428096000	63
18	6402373705728000	54
19	121645100408832000	45
20	2432902008176640000	54

Smallest Non-Niven Factorial

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Smallest Non-Niven Factorial

- In 1980, Kennedy, Goodman, and Best discovered for $n = 1, 2, \ldots, 431$, $n!$ is Niven.
- However, 432! is not Niven. This is because

$$s(432!) = 3897 = 9 \cdot 433 \text{ and } 433 \text{ is prime.}$$

Number of Niven Numbers

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Let x be a real number and $N(x)$ be the number of Niven numbers $\leq x$.

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Definition

Let x be a real number and $N(x)$ be the number of Niven numbers $\leq x$.

Here is a table of x and $N(x)$.

x	1	10	100	1000	10^4	10^5	10^6	10^7
$N(x)$	1	10	33	213	1538	11872	95428	806095

Natural Density of Niven Numbers

- In 1984, Cooper and Kennedy showed that the natural density of the Niven numbers is zero. That is,

Theorem

$$\lim_{x \to \infty} \frac{N(x)}{x} = 0.$$

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Theorem

$$\lim_{x \to \infty} \frac{N(x)}{x} = 0.$$

- The proof of the theorem relied on Chebyshev's inequality and the following lemma.

Statistics of Digital Sums

Lemma

Let n be a positive integer and consider the set

$$S = \{s(0),\ s(1),\ s(2),\ \ldots, s(10^n - 1)\}.$$

Then the mean and standard deviation of $s(n)$ over the set S is

$$\mu = 4.5n \text{ and } \sigma = \sqrt{8.25n}.$$

Proof of Lemma

- The lemma can be proved by considering a random experiment consisting of throwing n ten-faced dice, where each of the ten faces is marked with one of the numbers 0, 1, 2, . . ., 9. The sample space associated with this experiment consists of 10^n points

$$\{(x_1,\ x_2,\ \ldots,\ x_n)|0 \leq x_i \leq 9\}.$$

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$$\{(x_1,\ x_2,\ \ldots,\ x_n) | 0 \leq x_i \leq 9\}.$$

- Each outcome represents the digits of a number in the interval $0 \leq x < 10^n$.

Proof of Lemma cont.

$$\frac{1}{10^n}\sum_{x=0}^{10^n-1} s(x) = \mu = \mathrm{E}(x_1 + x_2 + \cdots + x_n) = n\mathrm{E}(x_1)$$

and

$$\frac{1}{10^n}\sum_{x=0}^{10^n-1} (s(x)-\mu)^2 = \sigma^2 = \mathrm{Var}(x_1 + x_2 + \cdots + x_n) = n\mathrm{Var}(x_1).$$

Proof of Lemma cont.

- But,

$$\mathrm{E}(x_1) = \frac{1}{10}(0+1+2+\cdots+9) = 4.5$$

and

$$\begin{aligned}\mathrm{Var}(x_1) &= \mathrm{E}(x_1^2) - (\mathrm{E}(x_1))^2 \\ &= \frac{1}{10}(0^2+1^2+2^2+\cdots+9^2) - (4.5)^2 = 8.25.\end{aligned}$$

Proof of Lemma cont.

- But,

$$\mathrm{E}(x_1) = \frac{1}{10}(0 + 1 + 2 + \cdots + 9) = 4.5$$

and

$$\begin{aligned}\mathrm{Var}(x_1) &= \mathrm{E}(x_1^2) - (\mathrm{E}(x_1))^2 \\ &= \frac{1}{10}(0^2 + 1^2 + 2^2 + \cdots + 9^2) - (4.5)^2 = 8.25.\end{aligned}$$

- Therefore,

$$\mu = 4.5n \ \text{ and } \ \sigma = \sqrt{8.25n}.$$

Lower and Upper Bounds of Niven Numbers

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Let $\epsilon > 0$ *be given. Then*

$$x^{1-\epsilon} \ll N(x) \ll \frac{x \log \log x}{\log x}.$$

Lower and Upper Bounds of Niven Numbers

- In 2003, DeKoninck and Doyon found upper and lower bounds for $N(x)$.

Theorem

Let $\epsilon > 0$ be given. Then

$$x^{1-\epsilon} \ll N(x) \ll \frac{x \log \log x}{\log x}.$$

- That is, given any $\epsilon > 0$, there exists a positive real number $x_0 = x_0(\epsilon)$ such that

$$N(x) > x^{1-\epsilon} \text{ for all } x \geq x_0.$$

Heuristic Asymptotic Formula for Niven Numbers

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- That is,

$$N(x) \sim c\frac{x}{\log x},$$

where

$$c = \frac{14}{27}\log 10$$

- Here, $f(x) \sim g(x)$ means that

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1.$$

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$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10$$

  is an example of 10 consecutive Niven numbers.

Consecutive Niven Numbers

- Question. How many consecutive Niven numbers are possible?
- It is not difficult to find sequences of consecutive Niven numbers. For example,

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10$$

  is an example of 10 consecutive Niven numbers.
- Sequences of 3 and 5 consecutive Niven numbers are

$$110,\ 111,\ 112; \text{ and}$$
$$131052,\ 131053,\ 131054,\ 131055,\ 131056; \text{ respectively.}$$

Consecutive Niven Numbers

To discuss consecutive Niven numbers, we introduce the idea of a decade and a century of numbers.

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Definition

Let n be a nonnegative integer. A decade is a set of numbers

$$\{10n,\ 10n+1,\ \ldots,\ 10n+9\}$$

and a century is a set of numbers

$$\{100n,\ 100n+1,\ \ldots,\ 100n+99\}.$$

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Let E denote the statement "odd numbers which have an even digital sum" and O denote the statement "odd numbers which have an odd digital sum."

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- To make another observation, we need the following definition.

Definition

Let E denote the statement "odd numbers which have an even digital sum" and O denote the statement "odd numbers which have an odd digital sum."

- Note that the ten decades in a century alternate either

$$\mathrm{O}, \mathrm{E}, \mathrm{O}, \mathrm{E}, \mathrm{O}, \mathrm{E}, \mathrm{O}, \mathrm{E}, \mathrm{O}, \mathrm{E} \text{ or } \mathrm{E}, \mathrm{O}, \mathrm{E}, \mathrm{O}, \mathrm{E}, \mathrm{O}, \mathrm{E}, \mathrm{O}, \mathrm{E}, \mathrm{O}.$$

Consecutive Niven Numbers

- Finally, note that in an E decade, none of the odd numbers can be Niven since their digital sum is even.

Consecutive Niven Numbers

- Finally, note that in an E decade, none of the odd numbers can be Niven since their digital sum is even.
- Thus, the only way to get more than 11 consecutive Niven numbers is to cross a century boundary where the decades between centuries would be

$$\ldots,\ \mathrm{E}, \mathrm{O}, \mathrm{E}, \mathrm{O} \mid \mathrm{O}, \mathrm{E}, \mathrm{O}, \mathrm{E},\ \ldots.$$

Consecutive Niven Numbers

- Hence, we cannot have more than 21 consecutive Niven numbers and if a list of 21 consecutive Niven numbers exists, it would have to begin with an even Niven number of the form
$$n \cdot 10^2 + 90,$$
where n is a nonnegative integer.

Consecutive Niven Numbers

- Hence, we cannot have more than 21 consecutive Niven numbers and if a list of 21 consecutive Niven numbers exists, it would have to begin with an even Niven number of the form
$$n \cdot 10^2 + 90,$$
where n is a nonnegative integer.
- For example,
$$2390,\ 2391,\ \ldots,\ 2399,\ 2400,\ 2401,\ \ldots,\ 2409,\ 2410$$
would be a possible example.

Consecutive Niven Numbers

- We will denote n_r as the concatenation of r n's in its decimal representation.

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- In 1993, Cooper and Kennedy stated and proved the following theorem.

Family of Consecutive Niven Numbers

Theorem

Let m be a nonnegative integer and let

$$\begin{aligned} a =& 4090669070187777592348077471447408839621564801 \\ & 2007115516094806249015486761744582584646124234 \\ & 1540855543641742325745294115007591954820126570 \\ & 087071005523266064292043054902370439430_{1120} \end{aligned}$$

and

$$\begin{aligned} b =& 2846362190166818204716429619770154544233311863 \\ & 4187301827478422658543387589306681088151446703 \\ & 2759507916140833155837906335537198825206802774 \\ & 843028314975502097292745955936059236215 69_{1119}0. \end{aligned}$$

Family of Consecutive Niven Numbers

Theorem

Finally, let

$$x = a_{3423103}0_m b.$$

Then

$$x,\ x+1,\ x+2,\ \ldots,\ x+19$$

is a sequence of 20 consecutive Niven numbers.

Theorem on Consecutive Niven Numbers

- Then we proved the following theorem.

Theorem

There does not exist a sequence of 21 consecutive Niven numbers.

b-Niven Numbers

Definition

Let $b \geq 2$ be an integer. A b-Niven number is a positive integer which is divisible by the sum of its base b digits.

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Let $b \geq 2$ be an integer. A b-Niven number is a positive integer which is divisible by the sum of its base b digits.

- In 1994, Grundman extended the Cooper and Kennedy result to show that for bases $b \geq 2$, there are $2b$ but not $2b + 1$ consecutive b-Niven numbers.

Niven Repunits

Next, we studied Niven repunits.

Definition

Let n be a positive integer. Then 1_n is a repunit.

- Question. Find a characterization of Niven repunits.

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Definition

Let n be a positive integer. Then 1_n is a repunit.

- Question. Find a characterization of Niven repunits.
- The first four Niven repunits are

$$1, \; 111, \; 111111111, \; \text{and } 1_{27}.$$

Characterization of Niven Repunits

Theorem

Let n and 10 be relatively prime. Denote the order of 10 $\pmod{n}$ by $e_n(10)$. Then the following statements are equivalent.

(1) 1_n *is a Niven repunit.*

(2) $10^n \equiv 1 \pmod{n}$.

(3) $n \equiv 0 \pmod{e_n(10)}$.

(4) $n \equiv 0 \pmod{e_p(10)}$ *for each prime factor p of n.*

Generating Niven Repunits

- For every nonnegative integer t, 1_{3^t} is a Niven repunit. This follows from the fact that $e_3(10) = 1$ and statement (4) of the theorem.

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- Using statement (4) of the theorem, we can construct all n such that 1_n is Niven by determining which primes p are such that every prime factor of $e_p(10)$ also satisfies the condition of statement (4).

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- Using statement (4) of the theorem, we can construct all n such that 1_n is Niven by determining which primes p are such that every prime factor of $e_p(10)$ also satisfies the condition of statement (4).
- Since $e_7(10) = 6$ has a factor of 2, it follows that no multiple of 7 can satisfy statement (4). That is, 1_{7m} can never be a Niven repunit.

Generating Niven Repunits

- The first prime larger than 3 that can be a factor of an n that satisfies statement (4) is 37. This follows because $e_{37}(10) = 3$ and, as stated above, 3 is a prime that must be a factor of every n that satisfies statement (4) of the theorem.

Generating Niven Repunits

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- The next two primes, after 37, which could possibly be factors of an n such that 1_n is Niven are 163 and 757 since

$$e_{163}(10) = 3^4 \text{ and } e_{757}(10) = 3^3.$$

Generating Niven Repunits

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- The next two primes, after 37, which could possibly be factors of an n such that 1_n is Niven are 163 and 757 since

$$e_{163}(10) = 3^4 \text{ and } e_{757}(10) = 3^3.$$

- The first column in the following table gives all primes, less than 50000, which could possibly be factors of an n that satisfies statement (4).

Table of Niven Repunits

prime p	$e_p(10)$
3	1
37	3
163	$81 = 3^4$
757	$27 = 3^3$
1999	$999 = (3^3)(37)$
5477	$1369 = 37^2$
8803	$1467 = (3^2)(163)$
9397	$81 = 3^4$
13627	$6813 = (3^2)(757)$
15649	$489 = (3)(163)$
36187	$18093 = (3)(37)(163)$
40879	757

Generators of Niven Repunits

The following table gives a list of generators of Niven repunits.

3
(3)(37)
(3^4) (163)
$(3^3)(757)$
$(3^3)(37)(1999)$
$(3)(37^2)(5477)$
$(3^4)(163)(8803)$
$(3^4)(9397)$
$(3^3)(757)(13627)$
(3)(163)(15649)

Generators of Niven Repunits

- The phrase, "... generators of Niven repunits ..." is used because
  increasing the exponents of any of the prime factors of the least common multiple of any collection chosen from the list given in the table above will be an n such that 1_n *is a Niven repunit*.

Generators of Niven Repunits

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  increasing the exponents of any of the prime factors of the least common multiple of any collection chosen from the list given in the table above will be an n such that 1_n *is a Niven repunit*.
- For example,

$$\mathrm{lcm}\left((3^4)(163),\ (3^3)(757),\ (3^3)(757)(13627)\right)$$
$$= (3^4)(163)(757)(13627).$$

- So,

$$3_{n_1}163_{n_2}757_{n_3}13627_{n_4}$$

  will be a Niven repunit for any $n_1 \geq 4$, $n_2 \geq 1$, $n_3 \geq 1$, and $n_4 \geq 1$.

Questions on Niven Numbers

- Question 1. Find necessary and sufficient conditions for Niven factorials.

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- Question 2. What powers of 2 are Niven numbers?

$$2^1,\ 2^2,\ 2^3,\ 2^9,\ 2^{36},\ 2^{85},\ 2^{176},\ 2^{194},\ 2^{200},\ 2^{375},\ 2^{1517},$$
$$2^{1573},\ 2^{3042},\ 2^{5953},\ 2^{6043},\ 2^{6109},\ 2^{12068},\ 2^{12104},\ \ldots$$

π-Sudoku

A *Sudoku* puzzle is a 9×9 grid that is partially filled with integers from 1 to 9 as *clues*.

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3				2		9		
	1				3			
		4						
			1			3	2	9
2	8			5				7
					9			4
	5		6			2		
		2					6	
7		6		4				5

Figure: π-Sudoku

Solution of π-Sudoku

The solution to a puzzle is a fully filled grid with no duplications in each row, column, and each of the nine 3×3 squares.

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3	7	8	4	2	6	9	5	1
6	1	5	9	8	3	4	7	2
9	2	4	7	1	5	8	3	6
5	4	7	1	6	8	3	2	9
2	8	9	3	5	4	6	1	7
1	6	3	2	7	9	5	8	4
8	5	1	6	9	7	2	4	3
4	9	2	5	3	1	7	6	8
7	3	6	8	4	2	1	9	5

Figure: Solution to π-Sudoku

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- A puzzle is considered valid if it can be solved uniquely.
- We also want to consider a well-constructed puzzle to be minimal, that is, removal of any clue results in multiple solutions.
- We shall study the structure and properties of Sudoku puzzles and establish some strategies for solving puzzles deterministically, i.e., without trial-and-error.

Sudoku Terminology

- We consider each row of nine cells as a *row block*, each column of nine cells as a *column block*, and each of the nine 3×3 cells as a *square block*.

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- We consider each row of nine cells as a *row block*, each column of nine cells as a *column block*, and each of the nine 3×3 cells as a *square block*.
- There are nine blocks of each type.
- Now, we may also consider every cell to be at the intersection of a row block, a column block, and a square block.

Sudoku Terminology

- We number the nine row blocks from top to bottom as

$$R_1,\ R_2, \ldots,\ R_9,$$

the nine column blocks from left to right as

$$C_1,\ C_2,\ \ldots,\ C_9,$$

and the nine square blocks in row-major order as

$$S_1,\ S_2,\ \ldots,\ S_9.$$

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and the nine square blocks in row-major order as

$$S_1,\ S_2,\ \ldots,\ S_9.$$

- We use c_{ij} to denote the cell in R_i and C_j.

Rule 1

Theorem (Rule 1)

Let c be an unfilled cell. If every number except N appears in at least one block of c, then the solution to c is N.

Application of Rule 1

We will show, using Rule 1, that cell c_{58} is a 1.

3				2		9		
	1				3			
		4						
			1			3	2	9
2	8			5			1	7
					9			4
	5		6			2		
		2					6	
7		6		4				5

Figure: π-Sudoku

Application of Rule 1

2, 3, 4, 7, 9 in S_6; 5, 8 in R_5; 6 in C_8. So, c_{58} is 1.

3				2		9		
	1				3			
		4						
			1			3	2	9
2	8			5			1	7
					9			4
	5		6			2		
		2					6	
7		6		4				5

Figure: Partial Solution to π-Sudoku

Another Application of Rule 1

We will show, using Rule 1, that cell c_{57} is a 6.

3				2		9		
	1				3			
		4						
			1			3	2	9
2	8			5		6	1	7
					9			4
	5		6			2		
		2					6	
7		6		4				5

Figure: Partial Solution to π-Sudoku

Another Application of Rule 1

1, 2, 3, 4, 7, 9 in S_6; 5, 8 in R_5. So, c_{57} is 6.

3				2		9		
	1				3			
		4						
			1			3	2	9
2	8			5		6	1	7
					9			4
	5		6			2		
		2					6	
7		6		4				5

Figure: Partial Solution to π-Sudoku

Rule 2

Theorem (Rule 2)

Let c be an unfilled cell in a square block S and let N be a number that does not appear in any blocks containing c. If, in S, every unfilled cell other than c lies in a row block or a column block that contains N implicitly or explicitly, then the solution to c is N.

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- Note that a similar Rule 2 can be written for a row block R or column block C.

Application of Rule 2

We will show, using Rule 2, that c_{32} is 2.

3				2		9		
	1				3			
	2	4						
			1			3	2	9
2	8	9	3	5	4	6	1	7
					9			4
	5		6			2		
		2					6	
7		6		4				5

Figure: Partial Solution π-Sudoku

Application of Rule 2

S_1 doesn't contain a 2. c_{21} and c_{31} in C_1 can't be 2. c_{13} and c_{23} in C_3 can't be 2. c_{12} in R_1 can't be 2. So, c_{32} is 2.

3				2		9		
	1				3			
	2	4						
			1			3	2	9
2	8	9	3	5	4	6	1	7
					9			4
	5		6			2		
		2					6	
7		6		4				5

Figure: Partial Solution π-Sudoku

Another Application of Rule 2

Again, we will use Rule 2 to show that c_{29} is 2.

3				2		9		
	1				3			2
	2	4						
			1			3	2	9
2	8	9	3	5	4	6	1	7
					9			4
	5		6			2		
		2					6	
7		6		4				5

Figure: Partial Solution to π-Sudoku

Another Application of Rule 2

S_3 doesn't contain a 2. c_{18} and c_{19} in R_1 can't be 2. c_{37}, c_{38}, and c_{39} in R_3 can't be 2. c_{27} in C_7 can't be 2. And c_{28} in C_8 can't be a 2. So, c_{29} is 2.

3				2		9		
	1				3			2
	2	4						
			1			3	2	9
2	8	9	3	5	4	6	1	7
					9			4
	5		6			2		
		2					6	
7		6		4				5

Partial Solution to π-Sudoku

Applying Rules 1 and 2 several times to the π-Sudoku, we reach the partially solved puzzle.

3				2		9		
	1				3			2
	2	4						
			1			3	2	9
2	8	9	3	5	4	6	1	7
			2		9			4
	5		6			2		
		2					6	
7		6		4	2	1		5

Figure: Partial Solution to π-Sudoku

Pigeonhole Principle

- We obtain the following rule from the Pigeonhole Principle.

Pigeonhole Principle

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Theorem (Pigeonhole Principle)

Let B be a block with n unfilled cells, and let N_1, N_2, . . ., N_m be numbers that do not appear in B, where $m < n$. If these m N_i's are possible solutions only to certain m cells in B, then these N_i's are the only possible solutions to these m cells.

Partial Solution to π-Sudoku

1 and 6 are not in S_3. 1 and 6 appear in C_7 and C_8. And c_{29} is filled. So, by the Pigeonhole Principle, 1 and 6 are the only solutions to c_{19} and c_{39}. So, c_{38} is 3.

3				2		9		61
	1				3			2
	2	4					3	61
			1			3	2	9
2	8	9	3	5	4	6	1	7
			2		9			4
	5		6			2		
		2					6	
7		6		4	2	1		5

Partial Solution to π-Sudoku

Using several more applications of Rules 1 and 2, we are left with the following puzzle.

3			4	2		9		1
	1		9		3	4		2
9	2	4	7	1			3	6
	4		1			3	2	9
2	8	9	3	5	4	6	1	7
1		3	2		9			4
8	5	1	6	9	7	2	4	3
4	9	2	5	3	1	7	6	8
7	3	6	8	4	2	1	9	5

Figure: Partial Solution to π-Sudoku

Another Pigeonhole Sudoku

	5	9			8	4		
	1			6			3	
	6		4	3	7			
	4						6	
						9	1	8
		8		2				
						6		
		5					8	7
				7	6		5	

Another Pigeonhole Sudoku cont.

After using Rules 1 and 2, we have the following partial solution.

3	5	9	2	1	8	4	7	6
	1			6		8	3	2
8	6	2	4	3	7	5	9	1
	4					2	6	
		6		5	4	9	1	8
		8	6	2		7	4	
						6	2	
6		5		4	2		8	7
				7	6		5	

Applying Pigeonhole Principle

In C_2, 3 and 9 are the only solutions to c_{62} and c_{82}. So, c_{54} is 3.

3	5	9	2	1	8	4	7	6
	1			6		8	3	2
8	6	2	4	3	7	5	9	1
	4					2	6	
		6	3	5	4	9	1	8
	93	8	6	2		7	4	
						6	2	
6	93	5		4	2		8	7
				7	6		5	

Definition of Alternating Cycle-of-N

- We shall define several path-like structures.

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- First we consider the adjacency between two unfilled cells. If N is a potential solution to $c_{i_1 j_1}$ and $c_{i_2 j_2}$ that are in the same block, then we say these two cells are *adjacent* and we write $c_{i_1 j_1} \leftrightarrow c_{i_2 j_2}$.

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- We shall define several path-like structures.
- First we consider the adjacency between two unfilled cells. If N is a potential solution to $c_{i_1 j_1}$ and $c_{i_2 j_2}$ that are in the same block, then we say these two cells are *adjacent* and we write $c_{i_1 j_1} \leftrightarrow c_{i_2 j_2}$.
- Furthermore, if N is a potential solution only to these two cells in the block, we call them *2-adjacent* and, when it becomes necessary, we write $c_{i_1 j_1} \Leftrightarrow c_{i_2 j_2}$.

Definition of Alternating Cycle-of-N

- Assume that N is a potential solution to m cells $c_{i_1 j_1}$, $c_{i_2 j_2}$, $\ldots$, $c_{i_m j_m}$. If cells $c_{i_t j_t}$ and $c_{i_{t+1} j_{t+1}}$ are adjacent for $1 \leq t \leq m-1$, then the m cells form a *walk-of-N* of length $m-1$.

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- We use $c_{i_1 j_1} \leftrightarrow c_{i_2 j_2} \leftrightarrow \cdots \leftrightarrow c_{i_m j_m}$ to denote this walk.

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- We use $c_{i_1 j_1} \leftrightarrow c_{i_2 j_2} \leftrightarrow \cdots \leftrightarrow c_{i_m j_m}$ to denote this walk.
- Again, we may use "$\Leftrightarrow$" instead of "$\leftrightarrow$" when it is applicable and necessary. A walk-of-N is *closed* if the "first" cell and the "last" cell are identical. A *path-of-N* is a walk-of-N with no repeated cells. A *cycle-of-N* is a closed walk-of-N in which there are no repeated cells.

Definition of Alternating Cycle-of-N

- Now, let $c_{i_1 j_1} \leftrightarrow c_{i_2 j_2} \leftrightarrow \cdots \leftrightarrow c_{i_m j_m} \leftrightarrow c_{i_1 j_1}$ be an cycle-of-N of length $m \geq 5$.

Definition of Alternating Cycle-of-N

- Now, let $c_{i_1 j_1} \leftrightarrow c_{i_2 j_2} \leftrightarrow \cdots \leftrightarrow c_{i_m j_m} \leftrightarrow c_{i_1 j_1}$ be an cycle-of-N of length $m \geq 5$.
- If $c_{i_{2t} j_{2t}} \Leftrightarrow c_{i_{2t+1} j_{2t+1}}$, then we call this cycle-of-N an *alternating cycle-of-N*. Namely, every other adjacency is definitely a 2-adjacency starting from the second one: $c_{i_2 j_2} \Leftrightarrow c_{i_3 j_3}$, $c_{i_4 j_4} \Leftrightarrow c_{i_5 j_5}$, $\ldots$.

Definition of Alternating Cycle-of-N

- Now, let $c_{i_1j_1} \leftrightarrow c_{i_2j_2} \leftrightarrow \cdots \leftrightarrow c_{i_mj_m} \leftrightarrow c_{i_1j_1}$ be an cycle-of-N of length $m \geq 5$.
- If $c_{i_{2t}j_{2t}} \Leftrightarrow c_{i_{2t+1}j_{2t+1}}$, then we call this cycle-of-N an *alternating cycle-of-N*. Namely, every other adjacency is definitely a 2-adjacency starting from the second one: $c_{i_2j_2} \Leftrightarrow c_{i_3j_3}$, $c_{i_4j_4} \Leftrightarrow c_{i_5j_5}$, $\ldots$.
- Note if the length m is odd, the first and the last adjacency do not need to be a 2-adjacency. We call $c_{i_1j_1}$ the *pivot* cell. The following theorem provides a tool in solving more difficult Sudoku puzzles.

Alternating Cycle-of-N Rule

Theorem (Alternating Cycle-of-N)

If an alternating cycle-of-N of odd length is formed when solving a Sudoku puzzle, then N is not the solution to the pivot cell.

Alternating Cycle-of-N Rule

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If an alternating cycle-of-N of odd length is formed when solving a Sudoku puzzle, then N is not the solution to the pivot cell.

We will not prove this theorem here. However, we will apply it to our partial solution to the π-Sudoku puzzle.

Partial Solution to π-Sudoku

Again, here is our partial solution to the π-Sudoku puzzle.

3			4	2		9		1
	1		9		3	4		2
9	2	4	7	1			3	6
	4		1			3	2	9
2	8	9	3	5	4	6	1	7
1		3	2		9			4
8	5	1	6	9	7	2	4	3
4	9	2	5	3	1	7	6	8
7	3	6	8	4	2	1	9	5

Figure: Partial Solution to π-Sudoku

Alternating Cycle-of-6

6 is a possible solution to c_{12}, c_{16}, c_{46}, c_{41}, and c_{21}. We denote this using a $6+$.

3	6+		4	2	6+	9		1
6+	1		9		3	4		2
9	2	4	7	1			3	6
6+	4		1		6+	3	2	9
2	8	9	3	5	4	6	1	7
1		3	2		9			4
8	5	1	6	9	7	2	4	3
4	9	2	5	3	1	7	6	8
7	3	6	8	4	2	1	9	5

Figure: Partial Solution to π-Sudoku

Alternating Cycle-of-6

$$c_{12} \leftrightarrow c_{16} \Leftrightarrow c_{46} \leftrightarrow c_{41} \Leftrightarrow c_{21} \leftrightarrow c_{12}.$$

is an alternating cycle-of-6 of length five. c_{12} is the pivot cell.

3	6+		4	2	6+	9		1
6+	1		9		3	4		2
9	2	4	7	1			3	6
6+	4		1		6+	3	2	9
2	8	9	3	5	4	6	1	7
1		3	2		9			4
8	5	1	6	9	7	2	4	3
4	9	2	5	3	1	7	6	8
7	3	6	8	4	2	1	9	5

Application of Alternating-Cycle-of-6

By the alternating cycle-of-6 rule, number 6 is not a solution to cell c_{12}. So, c_{12} is 7.

3	7		4	2	6+	9		1
6+	1		9		3	4		2
9	2	4	7	1			3	6
6+	4		1		6+	3	2	9
2	8	9	3	5	4	6	1	7
1		3	2		9			4
8	5	1	6	9	7	2	4	3
4	9	2	5	3	1	7	6	8
7	3	6	8	4	2	1	9	5

Figure: Partial Solution to π-Sudoku

Solution to π-Sudoku

We can finish the solution to the Sudoku puzzle by using Rule 1 several times.

3	7	8	4	2	6	9	5	1
6	1	5	9	8	3	4	7	2
9	2	4	7	1	5	8	3	6
5	4	7	1	6	8	3	2	9
2	8	9	3	5	4	6	1	7
1	6	3	2	7	9	5	8	4
8	5	1	6	9	7	2	4	3
4	9	2	5	3	1	7	6	8
7	3	6	8	4	2	1	9	5

Figure: Solution to π-Sudoku

Another Alternating Cycle Sudoku

			9					
1	6	9			2			
		7					3	
					7			
2				4				8
				3			7	
8								
	4			6				7
		1		5	3		4	2

Another Alternating Cycle Sudoku cont.

After using Rules 1 and 2, we have the following partial solution.

4			9	7	5			6
1	6	9	3	8	2	7	5	4
5		7	4	1	6		3	9
					7			
2	7			4				8
				3	8		7	
8		6	7		4			
3	4			6				7
7	9	1	8	5	3	6	4	2

Alternating Cycle-of-2

2 is a possible solution to c_{13}, c_{18}, c_{48}, c_{45}, c_{75}, c_{72}, and c_{83}. We denote this by $2+$.

4		2+	9	7	5		2+	6
1	6	9	3	8	2	7	5	4
5		7	4	1	6		3	9
				2+	7		2+	
2	7			4				8
				3	8		7	
8	2+	6	7	2+	4			
3	4	2+		6				7
7	9	1	8	5	3	6	4	2

Alternating Cycle-of-2

$$c_{13} \leftrightarrow c_{18} \Leftrightarrow c_{48} \leftrightarrow c_{45} \Leftrightarrow c_{75} \leftrightarrow c_{72} \Leftrightarrow c_{83} \leftrightarrow c_{13}.$$

is an alternating cycle-of-2 of length seven. c_{13} is the pivot cell.

4		2+	9	7	5		2+	6
1	6	9	3	8	2	7	5	4
5		7	4	1	6		3	9
				2+	7		2+	
2	7			4				8
				3	8		7	
8	2+	6	7	2+	4			
3	4	2+		6				7
7	9	1	8	5	3	6	4	2

Applying Alternating Cycle-of-2

By the alternating cycle-of-2 rule, number 2 is not a solution to cell c_{13}. So, c_{83} is 2.

4		2+	9	7	5		2+	6
1	6	9	3	8	2	7	5	4
5		7	4	1	6		3	9
				2+	7		2+	
2	7			4				8
				3	8		7	
8	2+	6	7	2+	4			
3	4	2+		6				7
7	9	1	8	5	3	6	4	2

Second π-Sudoku

Here are two other Sudoku puzzles. Both of these Sudoku puzzles can be solved using the alternating cycle rule.

3				4			2	
	1			9	2	8		3
		4						6
			1				5	
9				5				8
	6				9			
						2		
1			5				6	
2		9		7	8			5

Figure: Second π-Sudoku

Third π-Sudoku

3			6			7	2	
	1			8				
		4					9	
6			1			4		
				5				
	2	5			9			6
						2		
4	9	7					6	
	6		7					5

Figure: Third π-Sudoku

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