Identities in the Spirit of Ramanujan's Amazing Identity

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Outline

In his “lost notebook”, Ramanujan stated the following amazing identity.

Ramanujan

If

$$\sum_{n\geq 0} a_n x^n = \frac{1+53x+9x^2}{1-82x-82x^2+x^3},$$

$$\sum_{n\geq 0} b_n x^n = \frac{2-26x-12x^2}{1-82x-82x^2+x^3},$$

$$\sum_{n\geq 0} c_n x^n = \frac{2+8x-10x^2}{1-82x-82x^2+x^3},$$

then

$$a_n^3 + b_n^3 = c_n^3 + (-1)^n.$$

Hirschhorn demonstrated that using the algebraic identity from the "lost notebook",

$$\begin{aligned}&(x^2+7xy-9y^2)^3+(2x^2-4xy+12y^2)^3\\=&(2x^2+10y^2)^3+(x^2-9xy-y^2)^3,\end{aligned}$$

Ramanujan could have proved his amazing identity.

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Chen gave an algorithm to produce similar algebraic identities and Ramanujan-like identities.

Our goal is to use this procedure to find explicit algebraic identities and Ramanujan-like identities.

For example, using the algebraic identity

$$(9x^2 + 45xy - 135y^2)^3 + (10x^2 - 20xy + 172y^2)^3$$
$$=(12x^2 + 12xy + 138y^2)^3 + (x^2 - 83xy + y^2)^3,$$

we can prove the following Ramanujan-like identity.

Ramanujan-like Identity

If

$$\sum_{n\geq 0} a_n x^n = \frac{9 + 3609x - 135x^2}{1 - 6888x + 6888x^2 - x^3},$$

$$\sum_{n\geq 0} b_n x^n = \frac{10 - 1478x + 172x^2}{1 - 6888x + 6888x^2 - x^3},$$

$$\sum_{n\geq 0} c_n x^n = \frac{12 + 1146x + 138x^2}{1 - 6888x + 6888x^2 - x^3},$$

then

$$a_n^3 + b_n^3 = c_n^3 + 1.$$

Outline

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Ramanujan-like Identity

Let

$$
\begin{aligned}
&(r_1x^2 + s_1xy + t_1y^2)^3 + (r_2x^2 + s_2xy + t_2y^2)^3 \\
=&(r_3x^2 + s_3xy + t_3y^2)^3 + (x^2 - s_4xy - t_4y^2)^3,
\end{aligned}
$$

be an algebraic identity in variables x and y and fixed integers for the r's, s's, and t's.

Ramanujan-like Identity

Then, if

$$\sum_{n\geq 0} a_n x^n = \frac{r_1 + (s_1 s_4 + t_1 - r_1 t_4)x - t_1 t_4 x^2}{1 - (s_4^2 + t_4)x - (s_4^2 t_4 + t_4^2)x^2 + t_4^3 x^3},$$

$$\sum_{n\geq 0} b_n x^n = \frac{r_2 + (s_2 s_4 + t_2 - r_2 t_4)x - t_2 t_4 x^2}{1 - (s_4^2 + t_4)x - (s_4^2 t_4 + t_4^2)x^2 + t_4^3 x^3},$$

$$\sum_{n\geq 0} c_n x^n = \frac{r_3 + (s_3 s_4 + t_3 - r_3 t_4)x - t_3 t_4 x^2}{1 - (s_4^2 + t_4)x - (s_4^2 t_4 + t_4^2)x^2 + t_4^3 x^3}$$

then

$$a_n^3 + b_n^3 = c_n^3 + (-t_4)^{3n}.$$

Proof. Let $w_0 = 0$, $w_1 = 1$, and for $n \geq 0$,

$$w_{n+2} = s_4 w_{n+1} + t_4 w_n.$$

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The generating function for the sequence $\{w_n\}$ is given by

$$w(x) = \sum_{n \geq 0} w_n x^n = \frac{x}{1 - s_4 x - t_4 x^2}.$$

Now, if $x = w_{n+1}$ and $y = w_n$, then

$$\begin{aligned} x^2 - s_4 xy - t_4 y^2 &= w_{n+1}^2 - s_4 w_{n+1} w_n - t_4 w_n^2 \\ &= w_{n+1}^2 - w_n(s_4 w_{n+1} + t_4 w_n) \\ &= w_{n+1}^2 - w_n w_{n+2} = (-t_4)^n. \end{aligned}$$

Now, if $x = w_{n+1}$ and $y = w_n$, then

$$\begin{aligned} x^2 - s_4 xy - t_4 y^2 &= w_{n+1}^2 - s_4 w_{n+1} w_n - t_4 w_n^2 \\ &= w_{n+1}^2 - w_n(s_4 w_{n+1} + t_4 w_n) \\ &= w_{n+1}^2 - w_n w_{n+2} = (-t_4)^n. \end{aligned}$$

The last equality can be proved by induction on n.

Now, let

$$
\begin{aligned}
a_n &= r_1 x^2 + s_1 xy + t_1 y^2 = r_1 w_{n+1}^2 + s_1 w_{n+1} w_n + t_1 w_n^2, \\
b_n &= r_2 x^2 + s_2 xy + t_2 y^2 = r_2 w_{n+1}^2 + s_2 w_{n+1} w_n + t_2 w_n^2, \\
c_n &= r_3 x^2 + s_3 xy + t_3 y^2 = r_3 w_{n+1}^2 + s_3 w_{n+1} w_n + t_3 w_n^2.
\end{aligned}
$$

Now, let

$$a_n = r_1 x^2 + s_1 xy + t_1 y^2 = r_1 w_{n+1}^2 + s_1 w_{n+1} w_n + t_1 w_n^2,$$
$$b_n = r_2 x^2 + s_2 xy + t_2 y^2 = r_2 w_{n+1}^2 + s_2 w_{n+1} w_n + t_2 w_n^2,$$
$$c_n = r_3 x^2 + s_3 xy + t_3 y^2 = r_3 w_{n+1}^2 + s_3 w_{n+1} w_n + t_3 w_n^2.$$

We can show that

$$a_n^3 + b_n^3 = c_n^3 + (-t_4)^{3n}.$$

But, using generating function techniques, we can show that

$$\sum_{n\geq 0} w_n^2 x^n = \frac{x - t_4 x^2}{1-(s_4^2+t_4)x-(s_4^2 t_4+t_4^2)x^2+t_4^3x^3},$$

$$\sum_{n\geq 0} w_{n+1}^2 x^n = \frac{1 - t_4 x}{1-(s_4^2+t_4)x-(s_4^2 t_4+t_4^2)x^2+t_4^3x^3},$$

$$\sum_{n\geq 0} w_n w_{n+1} x^n = \frac{s_4 x}{1-(s_4^2+t_4)x-(s_4^2 t_4+t_4^2)x^2+t_4^3x^3}.$$

Hence,

$$\sum_{n\geq 0} a_n x^n = \frac{r_1 + (s_1 s_4 + t_1 - r_1 t_4)x - t_1 t_4 x^2}{1 - (s_4^2 + t_4)x - (s_4^2 t_4 + t_4^2)x^2 + t_4 x^3},$$

$$\sum_{n\geq 0} b_n x^n = \frac{r_2 + (s_2 s_4 + t_2 - r_2 t_4)x - t_2 t_4 x^2}{1 - (s_4^2 + t_4)x - (s_4^2 t_4 + t_4^2)x^2 + t_4^3 x^3},$$

$$\sum_{n\geq 0} c_n x^n = \frac{r_3 + (s_3 s_4 + t_3 - r_3 t_4)x - t_3 t_4 x^2}{1 - (s_4^2 + t_4)x - (s_4^2 t_4 + t_4^2)x^2 + t_4^3 x^3},$$

and the proof is complete.

Outline

Procedure to Search for Third Power Algebraic Identities.

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$$t_1^3 + t_2^3 = t_3^3 - t_4^3.$$

Also, select a range of integer values for s_1 and s_2 to search.

a. For each t_1, t_2, t_3, t_4, s_1, and s_2, compute s_3 and s_4 using the equations

$$s_3 = \frac{s_1 t_1^2 + s_2 t_2^2 + r_1^2 s_1 t_4^2 + r_2^2 s_2 t_4^2}{r_3^2 t_4^2 + t_3^2},$$
$$s_4 = r_3^2 s_3 - r_1^2 s_1 - r_2^2 s_2.$$

Make sure these constants can be computed and that they are integers.

b. Check the following conditions.

$$3r_1 t_1^2 + 3s_1^2 t_1 + 3r_2 t_2^2 + 3s_2^2 t_2 = 3r_3 t_3^2 + 3s_3^2 t_3 + 3t_4^2 - 3s_4^2 t_4,$$
$$6r_1 s_1 t_1 + s_1^3 + 6r_2 s_2 t_2 + s_2^3 = 6r_3 s_3 t_3 + s_3^3 + 6s_4 t_4 - s_4^3,$$
$$3r_1^2 t_1 + 3r_1 s_1^2 + 3r_2^2 t_2 + 3r_2 s_2^2 = 3r_3^2 t_3 + 3r_3 s_3^2 - 3t_4 + 3s_4^2.$$

b. Check the following conditions.

$$3r_1t_1^2 + 3s_1^2t_1 + 3r_2t_2^2 + 3s_2^2t_2 = 3r_3t_3^2 + 3s_3^2t_3 + 3t_4^2 - 3s_4^2t_4,$$
$$6r_1s_1t_1 + s_1^3 + 6r_2s_2t_2 + s_2^3 = 6r_3s_3t_3 + s_3^3 + 6s_4t_4 - s_4^3,$$
$$3r_1^2t_1 + 3r_1s_1^2 + 3r_2^2t_2 + 3r_2s_2^2 = 3r_3^2t_3 + 3r_3s_3^2 - 3t_4 + 3s_4^2.$$

c. If all the above conditions are satisfied (every equation is true), the resulting collection of r's, s's, and t's form an algebraic identity.

For Step 1, we first want r_1, r_2 and r_3 to be positive integers that satisfy

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Other trivial values of r_1, r_2, and r_3 are $r_1 = 1$ and $r_2 = r_3 = r$, where r is a positive integer.

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Other values of r_1, r_2, and r_3 ($r_1 < r_2$ and $r_2 \neq r_3$) can be found in the following table.

$$r_1^3 + r_2^3 = r_3^3 + 1.$$

r_1	r_2	r_3
9	10	12
64	94	103
73	144	150
135	235	249
244	729	738
334	438	495
368	1537	1544
577	2304	2316
1010	1897	1988
1033	1738	1852
1126	5625	5640
1945	11664	11682

For Step 2, we want integers t_1, t_2, t_3, and t_4 such that

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In the spirit of Ramanujan, we assume $t_4 = \pm 1$. We also wanted to find nontrivial t's. Some values of the t's can be found in the following table.

$$t_1^3 + t_2^3 = t_3^3 - t_4^3.$$

t_1	t_2	t_3	t_4
-9	6	-8	1
6	-9	-8	1
-9	8	-6	1
8	-9	-6	1
-8	-6	-9	-1
-6	-8	-9	-1
-8	9	6	-1
9	-8	6	-1
-6	9	8	-1
9	-6	8	-1
6	8	9	1
8	6	9	1

$$t_1^3 + t_2^3 = t_3^3 - t_4^3.$$

t_1	t_2	t_3	t_4
-12	9	-10	-1
9	-12	-10	-1
-12	10	-9	-1
10	-12	-9	-1
-10	-9	-12	1
-9	-10	-12	1
-10	12	9	1
12	-10	9	1
-9	12	10	1
12	-9	10	1
9	10	12	-1
10	9	12	-1

$$t_1^3 + t_2^3 = t_3^3 - t_4^3.$$

t_1	t_2	t_3	t_4
-103	64	-94	-1
64	-103	-94	-1
-103	94	-64	-1
94	-103	-64	-1
-94	-64	-103	1
-64	-94	-103	1
-94	103	64	1
103	-94	64	1
-64	103	94	1
103	-64	94	1
64	94	103	-1
94	64	103	-1

Also, for step 2, we determined some range to search in for s_1 and s_2. For these we typically would try something like integers between -1500 and 1500. The bounds on s_1 and s_2 vary depending on the speed of the search.

Outline

$$
\begin{aligned}
&(r_1x^2+s_1xy+t_1y^2)^3+(r_2x^2+s_2xy+t_2y^2)^3\\
=&(r_3x^2+s_3xy+t_3y^2)^3+(x^2-s_4xy-t_4y^2)^3,
\end{aligned}
$$

$$\begin{aligned}&(r_1x^2 + s_1xy + t_1y^2)^3 + (r_2x^2 + s_2xy + t_2y^2)^3\\=&(r_3x^2 + s_3xy + t_3y^2)^3 + (x^2 - s_4xy - t_4y^2)^3,\end{aligned}$$

We found the following results. The constants in each row of the following table satisfy the equation. We include the leading coefficient of 1 in the last trinomial. Recall that the form of the last trinomial is $x^2 - s_4xy - t_4y^2$.

r_1, s_1, t_1	r_2, s_2, t_2	r_3, s_3, t_3	1, s_4, t_4
1,556,-65601	2,-364,83802	2,-36,67402	1,756,1
1,61,-791	2,-40,1010	2,-4,812	1,83,-1
1,7,-9	2,-4,12	2,0,10	1,9,1
1,-25,135	2,-32,138	2,-36,172	1,9,1
1,-227,11161	2,-292,11468	2,-328,14258	1,83,-1
9,412,-11161	10,-180,14258	12,112,11468	1,756,1
9,-126,3753	10,236,-3230	12,96,2676	1,430,-1
9,45,-135	10,-20,172	12,12,138	1,83,-1
9,-169,791	10,-180,812	12,-220,1010	1,9,1
9,-1539,65601	10,-1640,67402	12,-2004,83802	1,83,-1
3753,-126,9	4528,200,-8	5262,84,6	1,430,-1
11161,3481,-791	11468,-1300,1010	14258,1292,812	1,6887,-
11161,412,-9	11468,-112,12	14258,180,10	1,756,1

The third row is the algebraic identity discovered by Ramanujan. This gives Ramanujan's amazing identity.

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The seventh row gives the algebraic identity

$$\begin{aligned}&(9x^2-126xy+3753y^2)^3+(10x^2+236xy-3230y^2)^3\\=&(12x^2+96xy+2676y^2)^3+(x^2-430y+y^2)^3.\end{aligned}$$

This produces the following Ramanujan-like identity.

Ramanujan-like Identity

If

$$\sum_{n\geq 0} a_n x^n = \frac{9 - 54172x + 3753x^2}{1 - 184899x + 184899x^2 - x^3},$$

$$\sum_{n\geq 0} b_n x^n = \frac{10 + 98260x - 3230x^2}{1 - 184899x + 184899x^2 - x^3},$$

$$\sum_{n\geq 0} c_n x^n = \frac{12 + 43968x + 2676x^2}{1 - 184899x + 184899x^2 - x^3},$$

then

$$a_n^3 + b_n^3 = c_n^3 + 1.$$

Outline

Let

$$\begin{aligned}&(x^2+s_1xy+t_1y^2)^4+(mx^2+s_2xy+t_2y^2)^4+(nx^2+s_3xy+t_3y^2)^4\\=&(mx^2+s_4xy+t_4y^2)^4+(nx^2+s_5xy+t_5y^2)^4+(x^2-s_6xy-t_6y^2)^4,\end{aligned}$$

be an algebraic identity in variables x and y and integer constants m, n, s_1, s_2, s_3, s_4, s_5, s_6, t_1, t_2, t_3, t_4, t_5, and t_6.

Then, if

$$\sum_{n\geq 0} a_n x^n = \frac{1 + (s_1 s_6 + t_1 - t_6)x - t_1 t_6 x^2}{1 - (s_6^2 + t_6)x - (s_6^2 t_6 + t_6^2)x^2 + t_6^3 x^3},$$

$$\sum_{n\geq 0} b_n x^n = \frac{m + (s_2 s_6 + t_2 - m t_6)x - t_2 t_6 x^2}{1 - (s_6^2 + t_6)x - (s_6^2 t_6 + t_6^2)x^2 + t_6^3 x^3},$$

$$\sum_{n\geq 0} c_n x^n = \frac{n + (s_3 s_6 + t_3 - n t_6)x - t_3 t_6 x^2}{1 - (s_6^2 + t_6)x - (s_6^2 t_6 + t_6^2)x^2 + t_6^3 x^3},$$

$$\sum_{n\geq 0} d_n x^n = \frac{m + (s_4 s_6 + t_4 - m t_6)x - t_4 t_6 x^2}{1 - (s_6^2 + t_6)x - (s_6^2 t_6 + t_6^2)x^2 + t_6^3 x^3},$$

$$\sum_{n\geq 0} e_n x^n = \frac{n + (s_5 s_6 + t_5 - n t_6)x - t_5 t_6 x^2}{1 - (s_6^2 + t_6)x - (s_6^2 t_6 + t_6^2)x^2 + t_6^3 x^3}$$

then

$$a_n^4 + b_n^4 + c_n^4 = d_n^4 + e_n^4 + (-t_6)^{4n}.$$

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2. Select a collection of sets of integers t_1, t_2, t_3, t_4, t_5, and $t_6 = \pm 1$ such that

$$t_1^4 + t_2^4 + t_3^4 = t_4^4 + t_5^4 + 1.$$

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$$t_1^4 + t_2^4 + t_3^4 = t_4^4 + t_5^4 + 1.$$

   Also, select a range of integer values for s_1, s_2, s_3, and s_4 to search.

a. For each t_1, t_2, t_3, t_4, t_5, t_6, s_1, s_2, s_3, and s_4, compute s_5 and s_6 using the equations

$$s_5 = \frac{s_1 t_1^3 + s_2 t_2^3 + s_3 t_3^3 - s_4 t_4^3 - s_1 t_6^3 - m^3 s_2 t_6^3 - n^3 s_3 t_6^3 + m^3 s_4 t_6^3}{n^3 t_6^3 + t_5^3}$$

$$s_6 = -s_1 - m^3 s_2 - n^3 s_3 + m^3 s_4 + n^3 s_5.$$

Make sure these constants can be computed and that they are integers.

b. Check the following conditions.

$$4t_1^3 + 6s_1^2t_1^2 + 4mt_2^3 + 6s_2^2t_2^2 + 4nt_3^3 + 6s_3^2t_3^2$$
$$= 4mt_4^3 + 6s_4^2t_4^2 + 4nt_5^3 + 6s_5^2t_5^2 - 4t_6^3 + 6s_6^2t_6^2,$$
$$12s_1t_1^2 + 4s_1^3t_1 + 12ms_2t_2^2 + 4s_2^3t_2 + 12ns_3t_3^2 + 4s_3^3t_3$$
$$= 12ms_4t_4^2 + 4s_4^3t_4 + 12ns_5t_5^2 + 4s_5^3t_5 - 12s_6t_6^2 + 4s_6^3t_6,$$
$$6t_1^2 + 12s_1^2t_1 + s_1^4 + 6m^2t_2^2 + 12ms_2^2t_2 + s_2^4 + 6n^2t_3^2 + 12ns_3^2t_3 + s$$
$$= 6m^2t_4^2 + 12ms_4^2t_4 + s_4^4 + 6n^2t_5^2 + 12ns_5^2t_5 + s_5^4 + 6t_6^2 - 12s_6^2t_6$$
$$12s_1t_1 + 4s_1^3 + 12m^2s_2t_2 + 4ms_2^3 + 12n^2s_3t_3 + 4ns_3^3$$
$$= 12m^2s_4t_4 + 4ms_4^3 + 12n^2s_5t_5 + 4ns_5^3 + 12s_6t_6 - 4s_6^3,$$
$$4t_1 + 6s_1^2 + 4m^3t_2 + 6m^2s_2^2 + 4n^3t_3 + 6n^2s_3^2$$
$$= 4m^3t_4 + 6m^2s_4^2 + 4n^3t_5 + 6n^2s_5^2 - 4t_6 + 6s_6^2.$$

c. If all the above conditions are satisfied (every equation is true), the resulting collection of m, n, s's, and t's form an algebraic identity.

$$t_1^4 + t_2^4 + t_3^4 = t_4^4 + t_5^4 + 1.$$

t_1	t_2	t_3	t_4	t_5
2	31	47	14	49
2	31	47	49	14
2	35	47	19	50
2	35	47	50	19
2	47	173	71	172
2	47	173	172	71
2	148	191	56	206
2	148	191	206	56
3	6	21	16	19
3	6	21	19	16
3	7	8	2	9
3	7	8	9	2
3	7	44	24	43

Outline

$$
\begin{aligned}
&(x^2 + s_1 xy + t_1 y^2)^4 + (mx^2 + s_2 xy + t_2 y^2)^4 + (nx^2 + s_3 xy + t_3 y^2)^4 \\
=&(mx^2 + s_4 xy + t_4 y^2)^4 + (nx^2 + s_5 xy + t_5 y^2)^4 + (x^2 - s_6 xy - t_6 y^2)^4,
\end{aligned}
$$

$$
\begin{aligned}
&(x^2 + s_1xy + t_1y^2)^4 + (mx^2 + s_2xy + t_2y^2)^4 + (nx^2 + s_3xy + t_3y^2)^4 \\
=&(mx^2 + s_4xy + t_4y^2)^4 + (nx^2 + s_5xy + t_5y^2)^4 + (x^2 - s_6xy - t_6y^2)^4,
\end{aligned}
$$

We found the following results. The constants in each row of the following table satisfy the equation. Again, we include the leading coefficient of 1 in the last trinomial. Recall that the form of the last trinomial is $x^2 - s_6xy - t_6y^2$.

$m = 1$ and $n = 2$

$1, s_1, t_1$	$1, s_2, t_2$	$2, s_3, t_3$	$1, s_4, t_4$	$2, s_5, t_5$	$1, s_6, t_6$
1,-4,4	1,-6,9	2,-10,13	1,-7,11	2,-10,12	1,3,-1
1,-3,4	1.-8,9	2,-11,13	1,-9,12	2,-11,11	1,2,1
1,-1,4	1,-2,9	2,-3,13	1,7,-12	2,-3,-11	1,10,-1
1,-4,5	1,-6,6	2,-10,11	1,-7,9	2,-10,10	1,3,-1
1,0,5	1,-2,6	2,-2,11	1,7,-9	2,-2,-10	1,9,1
1,-4,5	1,-5,6	2,-9,11	1,-7,10	2,-9,9	1,2,1
1,-5,6	1,-10,23	2,-15,29	1,-11,26	2,-15,27	1,4,-1
1,-4,6	1,-12,23	2,-16,29	1,-13,27	2,-16,26	1,3,1
1,0,6	1,-4,23	2,-4,29	1,11,-27	2,-4,-26	1,15,-1
1,-6,7	1,-7,14	2,-13,21	1,-9,18	2,-13,19	1,4,-1
1,-4,7	1,-12,14	2,-16,21	1,-13,19	2,-16,18	1,3,1
1,-4,7	1,0,14	2,-4,21	1,9,-19	2,-4,-18	1,13,-1
1,-7,8	1,-6,11	2,-13,19	1,-9,16	2,-13,17	1,4,-1

$m = 2$ and $n = 3$

1, s_1, t_1	2, s_2, t_2	3, s_3, t_3	2, s_4, t_4	3, s_5, t_5	1, s_6, t_6
1,-1,7	2,-2,14	3,-3,21	2,10,-19	3,-6,-18	1,16,-1
1,-8,8	2,-10,11	3,-18,19	2,-14,16	3,-17,17	1,3,-1
1,0,8	2,-2,11	3,-2,19	2,10,-16	3,-5,-17	1,15,1
1,-7,8	2,-9,11	3,-16,19	2,-13,17	3,-15,16	1,2,1
1,-8,10	2,-12,19	3,-20,29	2,-16,26	3,-19,25	1,3,1
1,-4,10	2,0,19	3,-4,29	2,12,-26	3,-7,-25	1,19,-1
1,-3,11	2,0,16	3,-3,27	2,12,-23	3,-6,-24	1,18,1
1,0,11	2,-4,39	3,-4,50	2,16,-46	3,-9,-45	1,25,-1
1,-8,13	2,-10,13	3,-18,26	2,-14,22	3,-17,23	1,3,-1
1,4,-13	2,0,-13	3,4,-26	2,4,-22	3,3,-23	1,1,1
1,-1,14	2,-3,41	3,-4,55	2,17,-49	3,-9,-50	1,26,1
1,-8,15	2,-12,19	3,-20,34	2,-16,30	3,-19,29	1,3,1

$m = 3$ and $n = 5$

$1, s_1, t_1$	$3, s_2, t_2$	$5, s_3, t_3$	$3, s_4, t_4$	$5, s_5, t_5$	$1, s_6, t_6$
1,-2,21	3,-4,41	5,6,-71	3,6,-69	5,4,-49	1,22,-1

The first row of the table for $m = 1$ and $n = 2$ gives the algebraic identity

$$(x^2 - 4xy + 4y^2)^4 + (x^2 - 6xy + 9y^2)^4 + (2x^2 - 10xy + 13y^2)^4$$
$$= (x^2 - 7xy + 11y^2)^4 + (2x^2 - 10xy + 12y^2)^4 + (x^2 - 3xy + y^2)^4.$$

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$$= (x^2 - 7xy + 11y^2)^4 + (2x^2 - 10xy + 12y^2)^4 + (x^2 - 3xy + y^2)^4.$$

This produces the following Ramanujan-like identity.

If

$$\sum_{n\geq 0} a_n x^n = \frac{1-7x+4x^2}{1-8x+8x^2-x^3},$$

$$\sum_{n\geq 0} b_n x^n = \frac{1-8x+9x^2}{1-8x+8x^2-x^3},$$

$$\sum_{n\geq 0} c_n x^n = \frac{2-15x+13x^2}{1-8x+8x^2-x^3},$$

$$\sum_{n\geq 0} d_n x^n = \frac{1-9x+11x^2}{1-8x+8x^2-x^3},$$

$$\sum_{n\geq 0} e_n x^n = \frac{2-16x+12x^2}{1-8x+8x^2-x^3},$$

then

$$a_n^4 + b_n^4 + c_n^4 = d_n^4 + e_n^4 + 1.$$

The row in the table for $m = 3$ and $n = 5$ gives the algebraic identity

$$(x^2 - 2xy + 21y^2)^4 + (3x^2 - 4xy + 41y^2)^4 + (5x^2 + 6xy - 71y^2)^4$$
$$= (3x^2 + 6xy - 69y^2)^4 + (5x^2 + 4xy - 49y^2)^4 + (x^2 - 22xy + y^2)^4.$$

The row in the table for $m = 3$ and $n = 5$ gives the algebraic identity

$$(x^2 - 2xy + 21y^2)^4 + (3x^2 - 4xy + 41y^2)^4 + (5x^2 + 6xy - 71y^2)^4$$
$$= (3x^2 + 6xy - 69y^2)^4 + (5x^2 + 4xy - 49y^2)^4 + (x^2 - 22xy + y^2)^4.$$

This produces the following Ramanujan-like identity.

If

$$\sum_{n\geq 0} a_n x^n = \frac{1-22x+21x^2}{1-483x+483x^2-x^3},$$

$$\sum_{n\geq 0} b_n x^n = \frac{3-44x+41x^2}{1-483x+483x^2-x^3},$$

$$\sum_{n\geq 0} c_n x^n = \frac{5+66x+71x^2}{1-483x+483x^2-x^3},$$

$$\sum_{n\geq 0} d_n x^n = \frac{3+66x+69x^2}{1-483x+483x^2-x^3},$$

$$\sum_{n\geq 0} e_n x^n = \frac{5+44x+49x^2}{1-483x+483x^2-x^3},$$

then

$$a_n^4 + b_n^4 + c_n^4 = d_n^4 + e_n^4 + 1.$$

Outline

1. In the third power case, we were unable to find any nontrivial algebraic identities with $r_1 = 1$ and $r_2 = r_3 = r$ where $r \geq 3$ is an integer. We would like to know if any algebraic identities exist and if so, what are they?

1. In the third power case, we were unable to find any nontrivial algebraic identities with $r_1 = 1$ and $r_2 = r_3 = r$ where $r \geq 3$ is an integer. We would like to know if any algebraic identities exist and if so, what are they?

2. We were unable to find any fourth power algebraic identities of the form

$$\begin{aligned}(r_1x^2 + s_1xy + t_1y^2)^4 &+ (r_2x^2 + s_2xy + t_2y^2)^4 \\ &+ (r_3x^2 + s_3xy + t_3y^2)^4 \\ =&(r_4x^2 + s_4xy + t_4y^2)^4 + (x^2 - s_5xy - t_5y^2)^4,\end{aligned}$$

   where the r's are positive integers and the s's and t's are nontrivial. Do such identities exist?

3. In the fourth power case, we found algebraic identities for every pair we tried when m is a positive integer and $n = m + 1$. Is this always true? In addition, is there any other algebraic identity where $n \neq m + 1$ other than the one we found where $m = 3$ and $n = 5$?

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