EVALUATION OF A FAMILY OF IMPROPER INTEGRALS

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It is easy to see that the improper integral

\[ I(q) = \int_{1}^{\infty} \frac{dx}{x(x^{q-1} + \cdots + x + 1)} \]

converges if \( q \geq 2 \). The purpose of this paper is to evaluate (1) when \( q \) is an integer.

It has been shown in [1], page 37, that

\[ \psi(a) - \psi(a - b) = \frac{\Gamma(a)}{\Gamma(b)} \sum_{n=1}^{\infty} \frac{\Gamma(b + n)}{n\Gamma(a + n)} \]

for \( Re(a) > Re(b) \geq 0 \). Equation (2) will be of particular interest when the parameters are specialized by letting \( a = 1 \) and \( b = \frac{1}{q} \) for \( q = 2, 3, 4, \ldots \). For then, (2) becomes

\[ \frac{\Gamma(1)}{\Gamma\left(\frac{1}{q}\right)} \sum_{n=1}^{\infty} \frac{\Gamma\left(\frac{1}{q} + n\right)}{n\Gamma(1 + n)} = \psi(1) - \psi\left(1 - \frac{1}{q}\right) , \]

and so

\[ \sum_{n=1}^{\infty} \frac{\left(\frac{1}{q}\right)_{n}}{n!n} = -\gamma - \psi\left(1 - \frac{1}{q}\right) , \]

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where $\gamma$ is the Euler-Mascheroni constant and

$$(r)_n = \frac{\Gamma(r + n)}{\Gamma(r)} = r(r + 1) \cdots (r + n - 1).$$

Now, from page 13 of [2], it is known that

$$\psi \left(1 - \frac{1}{q}\right) = -\gamma - \ln q - \frac{\pi}{2} cot \left(\frac{q - 1}{q}\right) \pi$$

$$+ \sum_{n=1}^{\left[\frac{q}{2}\right]} \cos \frac{2n(q - 1)\pi}{q} \ln \left(2 - 2\cos \frac{2n\pi}{q}\right),$$

where the prime attached to the summation index indicates that if $q$ is even then the last term in the sum is taken at half its weight.

Substituting (4) into (3) and simplifying gives

$$\sum_{n=1}^{\infty} \frac{1(1 + q)(1 + 2q) \cdots (1 + (n-1)q)}{n! \; q^n} = \ln q$$

$$+ \frac{\pi}{2} cot \left(\frac{q - 1}{q}\right) \pi - 2 \sum_{n=1}^{\left[\frac{q}{2}\right]} \cos \frac{2n(q - 1)\pi}{q} \ln \left(2 \sin \frac{n\pi}{q}\right).$$

Since

$$(1 - u)^{-\frac{1}{q}} = \sum_{n=0}^{\infty} \left(-\frac{1}{q}\right)^n (-u)^n \quad for \quad -1 \leq u < 1,$$
In (6), let
\[ x = (1 - u)^{-\frac{1}{q}}. \]

Then
\[ u = 1 - x^{-q}, \quad du = q x^{-q-1} dx, \]

and so
\[
\int_0^1 \frac{(1 - u)^{-\frac{1}{q}} - 1}{u} \, du = \sum_{n=1}^{\infty} \frac{(-\frac{1}{q})}{n} (-1)^n \int_0^1 u^{n-1} \, du
\]

\[
= \sum_{n=1}^{\infty} \frac{(-1)^n (-\frac{1}{q})}{n} \frac{(-\frac{1}{q} - 1) (-\frac{1}{q} - 2) \cdots (-\frac{1}{q} - n + 1)}{n!} \frac{1}{n}
\]

\[
= \sum_{n=1}^{\infty} \frac{1(1 + q)(1 + 2q) \cdots (1 + (n-1)q)}{n! q^n n}.
\]

Simplifying (7) and using (5) yields
\[
I(q) = \frac{1}{q} \left( \ln q + \frac{\pi}{2} \cot \left(\frac{q-1}{q} \pi\right) \right)
\]

\[
- 2 \sum_{n=1}^{\lfloor \frac{q}{2} \rfloor} \cos \frac{2n(q - 1)\pi}{q} \ln \left( 2 \left| \sin \frac{n\pi}{q} \right| \right).
\]
References
