ALMOST LOCALLY CONNECTED(SO) SPACES

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A topological space M is almost locally connected(SO) at a point p iff for any open subset U of M containing p and the component C of U containing p, $\overline{C} \cap \partial U \neq \emptyset$ and C is nonclosed [1]. In this paper, we discuss the structure and properties of almost locally connected(SO) spaces and some applications. Throughout this paper, we assume all the spaces are topological and Hausdorff.

Definition. A space M is *almost locally connected(SO)* if M is almost locally connected(SO) at every point.

Theorem 1. The following conditions are equivalent in the space M.

(1) M is almost locally connected(SO) at p.

(2) If U is an open subset of M containing p, there exists a nonclosed connected subset V of U containing p such that $\overline{V} \cap \partial U \neq \emptyset$.

(3) $\overline{C} \cap \overline{U} \neq \emptyset$.

(4) $\partial C \cap \partial U \neq \emptyset$ where U is an open subset containing p, C is the component of p in U and C is nonclosed.

Proof.

(1) $\Rightarrow$ (2) Let V be the component of U containing p. Then V is nonclosed and $\overline{V} \cap \partial U \neq \emptyset$.

(2) $\Rightarrow$ (3) If C is closed then $\overline{C} \cap \partial U = \emptyset$ and hence $\overline{V} \cap \partial U = \emptyset$ for every nonclosed connected subset V of U containing p, a contradiction.

(3) $\Rightarrow$ (4) $\overline{C} \cap \overline{U} \neq \emptyset$ and $C \subset U$ imply $\partial C \cap \partial U \neq \emptyset$.

(4) $\Rightarrow$ (1) This follows immediately from the definition.

Theorem 2. If M is almost locally connected(SO) at p and U is an open subset of M containing p, then U is almost locally connected(SO) at p.

Proof. If V is an open subset of U containing p, then V is open in M. Consequently, there exists a nonclosed connected subset G of V containing p such that $\overline{G} \cap \partial V \neq \emptyset$, and the theorem follows.

Theorem 3. M is almost locally connected(SO) iff for every subset

A of M and every component C of $M - A$, $\overline{C} \cap \partial A \neq \emptyset$.

Proof. "$\Rightarrow$" Suppose $\overline{C} \cap \partial A = \emptyset$. Since C is a subset of $M - A$, $\overline{C} \cap \overline{A} = \emptyset$. Hence, $\overline{C} \subset M - \overline{A}$. Now, M is almost locally connected(SO), so according to Theorem 2 there exists a nonclosed component K of $M - \overline{A}$ such that $\overline{K} \cap \partial(M - \overline{A}) \neq \emptyset$. By the definition of component, $C \subset K$. On the other hand, C is a component of $M - A$ and $M - \overline{A} \subset M - A$. Therefore, $K \subset C$ and hence C = K. It follows that $\overline{K} \subset M - \overline{A}$ and $\overline{K} \cap \partial(M - \overline{A}) \neq \emptyset$, a contradiction. Therefore, $\overline{C} \cap \partial A \neq \emptyset$.

"$\Leftarrow$" Suppose M is not almost locally connected(SO). Then there exists an open subset U of M and a nonclosed component C of M such that $\overline{C} \cap \partial U = \emptyset$. If $A = M - U$, then C is a component of $M - A$ and $\overline{C} \cap \partial A = \emptyset$, a contradiction.

Theorem 4. Let M be almost locally connected(SO) at p and let h be a homeomorphism from M to the space N. Then N is almost locally connected(SO) at h(p).

Proof. Let U be an open subset of N containing h(p). Then there exists an open set O in M such that $p \in O$ and $h(O) = U$.

Since M is almost locally connected(SO) at p, there is a nonclosed connected subset G of O such that $p \epsilon G$ and $\overline{G} \cap \partial O \neq \emptyset$. Since h is a homeomorphism, h(G) is nonclosed and connected.

Suppose $\overline{h(G)} \cap \partial U = \emptyset$. Then $h(\overline{G}) = \overline{h(G)} \subset U = h(O)$. This implies $\overline{G} \subset O$, which contradicts $\overline{G} \cap \partial O \neq \emptyset$. Therefore, $\overline{h(G)} \cap \partial U \neq \emptyset$ and N is almost locally connected(SO) at h(p).

Theorem 5. If M is connected and locally connected at p, then M is almost locally connected(SO) at p.

Proof. Theorem 5 follows from the fact that M is locally connected at p and no proper subset of M is both open and closed.

The following example shows that connectedness is necessary in Theorem 5.

Example 1. Let M be a nonempty finite set with the discrete topology. Then M is locally connected but M is not almost locally connected(SO) at any point in M.

Theorem 6. If M is almost locally connected(SO) and U is an open subset of M such that ∂U is connected, then $\overline{U}$ is connected.

Proof. Since

$$\overline{U} = \bigcup \{\overline{C} \mid C \ is \ a \ component \ of \ U \ and \ \overline{C} \cap \partial U \neq \emptyset\},$$

$\overline{U}$ is connected.

The following theorem is well known.

Theorem 7. If C is a component of the compact set X in a topological space, then each open set containing C contains an open set containing C whose boundary does not intersect X.

Theorem 8. If M is connected and locally compact at p, then M is almost locally connected(SO) at p.

Proof. Let U be an open set containing p such that $\overline{U}$ is compact. Let C be the component of U containing p. Suppose $\overline{C} \cap \partial U = \emptyset$. Then C is closed in U. Since $\overline{C}$ and ∂U are compact, there exist disjoint sets A and B relatively open in $\overline{U}$ such that $\overline{C} \subset A$ and $\partial U \subset B$. It follows that $\overline{C} \subset A$ and $\partial U \cap A = \emptyset$ so $A \subset U$. Applying Theorem 7 to $\overline{A}$, there exists an open set V such that $C \subset V \subset A$ and $\partial V \cap \overline{A} = \emptyset$. Hence $\partial V = \emptyset$. Consequently, V is both open and closed in M, which contradicts the fact that M is connected.

Definition. A set K in a Hausdorff, topological space M is called a *continuum* if K is compact and connected. A set K is called a *generalized continuum* if it is locally compact and connected.

Corollary. Every generalized continuum and thus every continuum is almost locally connected(SO).

Example 2. Let

$$M = \{(0,0)\} \cup \{(x,y) \mid 0 < x \leq 1 \; and \; y = sin\,(1/x)\}.$$

M is connected, but M is neither locally compact nor almost locally connected(SO) at (0,0).

Examples 1 and 2 illustrate that both connectedness and local compactness are necessary for Theorem 8. The following example illustrates that the converses of both Theorems 5 and 8 are not true even if the space is connected.

Example 3. Let

$$M = \{(0,y) \mid 0 \leq y \leq 1\} \cup \{(x,y) \mid 0 < x \leq 1, \; y = sin(1/x)\}.$$

M is connected and almost locally connected(SO) at (0,0), but M is neither locally compact nor locally connected at (0,0).

The following definition can be found in [2].

Definition. The connected set M is *semilocally connected at a point p* if each open set containing p contains an open set V containing p such that $M - V$ has at most a finite number of components.

Example 1 shows that semilocal connectedness does not imply almost local connectedness(SO).

Jones [3] established the following definitions and theorem.

Definition. A space M is *aposyndetic at a point p with respect to a point q* if there is a closed connected set H such that p is in the interior of H and H is a subset of $M - \{q\}$.

Definition. The space M is *aposyndetic at a point p* if it is aposyndetic at p with respect to q for each q in $M - \{p\}$.

Definition. The space M is *aposyndetic* if it is aposyndetic at every point.

Theorem 9. If M is semilocally connected, then M is aposyndetic.

Example 4. Let

$$M = \{(x,y) \mid y = sin(1/x),\ x \neq 0\} \cup \{(0,y) \mid 0 \leq y \leq 1\} \cup \{(-1/n,\ 0) \mid n = 1,2,3,\ldots\}\ .$$

M is almost locally connected(SO), but not aposyndetic at (0,0).

Definition. A *cut point* of a connected set M is a point p of M such that $M - \{p\}$ is disconnected.

Definition. p is an *end point* of a connected set M if each open set containing p contains an open set containing p whose boundary is degenerate.

Definition. A point r of a connected set M *separates points p and q in M* if $M - \{r\}$ is the union of two separated sets, one containing p and the other containing q.

Definition. Two points p and q of a connected set M are said to be *conjugate* in M if no point of M separates p and q in M.

Whyburn [2] first established the following decomposition theorem in cyclic element theory.

Theorem 10. If M is a connected, locally compact metric space and p is neither a cut point nor an end point of M, then there exists a point of M other than p which is conjugate to p.

Theorems 11 and 12 are similar results obtained by B. Lehman [4] and D. John [5] respectively.

Theorem 11. If M is a connected, locally compact, locally connected topological space and p is neither a cut point nor an end point of M, then there exists a point of M other than p which is conjugate to p.

Theorem 12. If M is a connected, locally compact topological space and p is neither a cut point nor an end point of M, then there exists a point of M other than p which is conjugate to p.

The statements in Theorems 10, 11 and 12 seem to suggest that local compactness plays a crucial part in the conclusion of these theorems. However, Theorem 14 below shows that the result still holds even without local compactness.

John [5] established the following theorem.

Theorem 13. If p is a non-cut point of a connected topological space M and M is semilocally connected at p, then for every open set U containing p there exists an open subset V of U containing p such that $M - V$ is connected.

Theorem 14. If M is a connected, almost locally connected(SO), semilocally connected topological space and p is neither a cut point nor an end point of M, then there exists a point of M other than p which is conjugate to p in M.

Proof. Suppose there is no point q in $M - \{p\}$ such that q is conjugate to p in M. Let U be an open set containing p. Then according to Theorem 13 there exists an open subset V of U containing p such that $M - V$ is connected. Let C be the component of V containing p. Since M is almost locally connected(SO), we have $\partial V \cap \overline{C} \neq \emptyset$. Let $z \epsilon \partial V \cap \overline{C}$. Since z is not conjugate to p in M, there exists y in M such that $M - \{y\} = H \cup K$, where H and K are separated and p is in H and z is in K. If y is not in C, then C is connected in $M - \{y\}$, which implies $C \cup \{z\}$ is a connected subset of $M - \{y\}$. It follows that $C \cup \{z\} \subset H$ or $C \cup \{z\} \subset K$, which implies $z \epsilon H$ or $p \epsilon K$. Either case is impossible.

Therefore, $y \epsilon C$. That is $M - V \subset K$. This implies $H \subset V$. Since H is open and $\partial H = \{y\}$, p is an end point and this contradicts the assumption.

References

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