SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcome by the editor.

1. *Proposed by Robert E. Kennedy and Curtis Cooper, Central Missouri State University, Warrensburg, Missouri.*

Let $n \geq 2$ and

$$\sum_{j=1}^{i+1} a_{ij} = 0 \quad \text{for} \quad i = 1,\ 2,\ \ldots,\ n-1\ .$$

Find the determinant of the nxn matrix

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 & \ldots & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \ldots & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & \ldots & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & a_{n-1,4} & \ldots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix} .$$

I. *Solution by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Using the given summation constraints, one can column reduce the given matrix, A , to a lower triangular matrix having main diagonal entries

$$a_{11}, a_{21}+a_{22}, a_{31}+a_{32}+a_{33}, \ldots , a_{n-1,1}+a_{n-1,2}+\cdots+a_{n-1,n-1}\ .$$

Using the constraints again, these diagonal entries can be replaced by

$$-a_{12}, -a_{23}, -a_{34}, \ldots , -a_{n-1,n}$$

respectively. Therefore,

$$\det(A) = (-1)^{n-1} a_{12} a_{23} \cdots a_{n-1,n} \ .$$

II. *Solution by Albert Dixon, The School of the Ozarks, Point Lookout, Missouri.*

Let

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 0 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

and let A be the matrix above. Then the matrix $C = AB$ (using the given constraints twice) is a lower triangular matrix whose entries along the diagonal can be written as

$$c_{i,i} = -a_{i,i+1} \quad \text{for} \quad 1 \le i \le (n-1) \quad \text{and} \quad c_{n,n} = 1 \ .$$

Finally, $\det(AB) = \det(A)\ \det(B)$ for all matrices A and B and since B is upper triangular while C is lower triangular, their respective determinants are simply the product of their diagonal entries. Therefore, we conclude that

$$\begin{aligned} \det(A) &= \det(A) \cdot 1 \\ &= \det(A) \cdot \det(B) \\ &= \det(AB) \\ &= \det(C) \\ &= (-1)^{n-1} \prod_{i=1}^{n-1} a_{i,i+1} \ . \end{aligned}$$

Also solved by Charles J. Allard, Polo R-VII High School, Polo, Missouri and the proposers.

2. *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

A pitcher faces a batter in an at-bat. Let b be the probability the pitch is a ball, w, the probability the batter swings and misses (wiffs), and f, the probability the batter hits a foul ball. What is the probability that the batter strikes out?

Solution by Charles J. Allard, Polo R-VII High School, Polo, Missouri.

The batter will strike out by wiffing after the count reaches two strikes. This ranges from wiffing on an 0-2 count to wiffing with a full count (3-2). This is further complicated by the fact that after two strikes the batter may foul off zero or more pitches without changing the count. Let N represent the event of fouling-off zero or more pitches with two strikes. The sum of the probabilities for event N is

$$1 + f + f^2 + \ \cdots \ = \frac{1}{1-f} \ .$$

Let S represent the event of either wiffing or fouling-off a pitch with less than two strikes. The probability that event S can happen is $w + f$. Furthermore, let B represent the event of taking a ball and W represent the event of wiffing.

A batter can strike out from an 0-2 count with the sequence of events

$$\begin{array}{cccc} 1 & 2 & & 3 \\ S & S & N & W \end{array}$$

The probability that the batter strikes out from an 0-2 count is

$$(w+f)^2 w \frac{1}{1-f} \ .$$

A batter can strike out from a 1-2 count with the 3 sequences of events

$$\begin{array}{cccccc}
1 & 2 & & 3 & & 4 \\
B & S & & S & N & W \\
S & B & & S & N & W \\
S & S & N & B & N & W
\end{array}$$

The probability that the batter strikes out from a 1-2 count is

$$(w+f)^2 bw \left(\frac{2}{1-f} + \frac{1}{(1-f)^2} \right) \ .$$

A batter can strike out from a 2-2 count with the 6 sequences of events

$$\begin{array}{cccccccc}
1 & 2 & & 3 & & 4 & & 5 \\
B & B & & S & & S & N & W \\
B & S & & B & & S & N & W \\
S & B & & B & & S & N & W \\
B & S & & S & N & B & N & W \\
S & B & & S & N & B & N & W \\
S & S & N & B & N & B & N & W
\end{array}$$

The probability that the batter strikes out from a 2-2 count is

$$(w+f)^2 b^2 w \left(\frac{3}{1-f} + \frac{2}{(1-f)^2} + \frac{1}{(1-f)^3} \right) \ .$$

Finally, a batter can strike out from a 3-2 count with the 10 sequences of events

1	2		3		4		5		6
S	B		B		B		S	N	W
B	S		B		B		S	N	W
B	B		S		B		S	N	W
B	B		B		S		S	N	W
S	B		B		S	N	B	N	W
B	S		B		S	N	B	N	W
B	B		S		S	N	B	N	W
S	B		S	N	B	N	B	N	W
B	S		S	N	B	N	B	N	W
S	S	N	B	N	B	N	B	N	W

The probability that the batter strikes out from a 3-2 count is

$$(w+f)^2 b^3 w \left(\frac{4}{1-f} + \frac{3}{(1-f)^2} + \frac{2}{(1-f)^3} + \frac{1}{(1-f)^4}\right) .$$

Adding these probabilities up, we get the probability that the batter strikes out is

$$\begin{aligned}
&\sum_{i=0}^{3} (w+f)^2 b^i w \sum_{j=0}^{i} \frac{i+1-j}{(1-f)^{j+1}} \\
&= \frac{(w+f)^2 w}{(1-f)^4} \sum_{i=0}^{3} \sum_{j=0}^{i} b^i (i+1-j)(1-f)^{3-j} \\
&= \frac{(w+f)^2 w}{(1-f)^4} \sum_{j=0}^{3} \sum_{i=j}^{3} b^i (i+1-j)(1-f)^{3-j} \\
&= \frac{(w+f)^2 w}{(1-f)^4} \sum_{j=0}^{3} \sum_{i=0}^{3-j} b^{i+j} (i+1)(1-f)^{3-j} \\
&= \frac{(w+f)^2 w}{1-f} \sum_{j=0}^{3} \left(\frac{b}{1-f}\right)^j \sum_{i=0}^{3-j} (i+1) b^i .
\end{aligned}$$

Also solved by the proposers.