ON DEFINING A HYPERBOLA

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The most common definitions of the hyperbola uses one of the following two approaches.

1. Given a fixed line ℓ and a fixed point F, if a point P moves such that the ratio $\frac{\text{distance of } P \text{ to } F}{\text{distance of } P \text{ to } \ell}$ is a constant greater than 1, then the locus of P is a hyperbola. The line ℓ and the point F act as directrix and focus respectively. The ratio is the eccentricity e of the hyperbola.

2. Given two fixed points F_1 and F_2, if a point P moves such that the difference $|PF_1 - PF_2|$ is a constant, then the locus of P is a hyperbola. The fixed points F_1 and F_2 are the foci of the curve.

In this paper we will show a different way of defining the hyperbola. We will also show that with a slight modification, this approach can be used to define an ellipse as well. We certainly do not claim that the definition given here is in any way better than, or superior to the classical definitions mentioned above. We leave it to the teacher to use this new approach as an 'observation', or give it as a homework assignment.

Let $F_1,\ F_2$ be two fixed points with distance $2c$ apart. These points will serve as the foci of the hyperbola. Let O be the midpoint of the line segment F_1F_2 (figure 1). Use O as the origin and the line F_1F_2 as x–axis, let (x, y) be the coordinates of the moving point P. Note that $F_1 = (-c, 0)$ and $F_2 = (c, 0)$.

<u>Claim 1</u>. If $PO^2 - PF_1 \cdot PF_2 = k$, the locus of P is a hyperbola.

<u>Proof</u>. From the geometry of the triangle PF_1F_2,

$$PF_1^2 + PF_2^2 = 2PO^2 + 2c^2, \text{ and } 2PO^2 - 2PF_1 \cdot PF_2 = 2k.$$

On adding, $PF_1^2 - 2PF_1 \cdot PF_2 + PF_2^2 = 2c^2 + 2k$, and so,

$$|PF_1 - PF_2| = \sqrt{2c^2 + 2k}.$$

Since $|PF_1 - PF_2| =$ constant, the locus of P is a hyperbola whose foci are F_1 and F_2. Also, if we assume $|PF_1 - PF_2| =$ constant and retrace the above steps, we see that the $PO^2 - PF_1 \cdot PF_2$ is a constant as well. This establishes our claim.

At this stage one might be curious about k. What is k, and what does it measure in physical terms? We know that when the hyperbola has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, and $|PF_1 - PF_2| = 2a$, where $2a$ is the distance between the vertices of the hyperbola.

On squaring $|PF_1 - PF_2| = 2a$, we have that

$$PF_1^2 - 2PF_1 \cdot PF_2 + PF_2^2 = 4a^2.$$

Using $PF_1^2 + PF_2^2 = 2PO^2 + 2c^2$, we obtain

$$2PO^2 + 2c^2 - 2PF_1 \cdot PF_2 = 4a^2,$$

or $PO^2 - PF_1 \cdot PF_2 = 2a^2 - c^2$.

Thus, $k = 2a^2 - c^2 = 2a^2 - (a^2 + b^2) = a^2 - b^2$, since $c^2 = a^2 + b^2$.

If $k = 0$ then $a = b$ and the hyperbola is rectangular. Its eccentricity $e = \sqrt{2}$, and its asymptotes are at right angles. In this case, if one so desires, one may choose the asymptotes as coordinate axes and the equation of the hyperbola takes the shape $xy =$ constant.

If $k \neq 0$, i.e. $a \neq b$, the value of

$$k = a^2 - b^2 = a^2 - a^2(e^2 - 1) = a^2(2 - e^2)$$

represents the deviation from rectangularity. If we choose $a = 1$, then $k = 2 - e^2$ measures how far the curve departs from being a rectangular hyperbola. If we can use the word 'crooked' for 'non–rectangular', then k measures how crooked is a crooked hyperbola.

Can an equation similar to $PO^2 - PF_1 \cdot PF_2 = k$ be used to define an ellipse? We present the following claim.

Claim 2. Given two fixed points F_1 and F_2 and O as the mid-point of the line segment F_1F_2, the locus of the point P for which $PO^2 + PF_1 \cdot PF_2 = k$ is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $k = a^2 + b^2$. This ellipse has center O and the foci $F_1,\ F_2$ lie on the x–axis.

Notice that now $k = a^2 + b^2$. Once again we are reminded that the difference between a formula for an ellipse and the corresponding formula for a hyperbola is the sign of the b^2 term. A formal proof is similar to that of Claim 1, and is omitted here.

In summary, we show the comparison of the various formulas for the two curves. The hyperbola has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Here $c^2 = a^2 + b^2$. If we use our new definition $PO^2 - PF_1 \cdot PF_2 = k$ to define the same hyperbola, then $k = a^2 - b^2 = a^2(2 - e^2)$. Furthermore, if we let $e = \sec\theta$ and use $c = ae$ we have that $k = a^2(2 - e^2) = \frac{c^2}{e^2}(2 - e^2) = c^2(2\cos^2\theta - 1) = c^2\cos 2\theta$. Note that 2θ is the angle between the asymptotes and that k

could be positive, zero, or negative. The value of k is zero precisely when $2\theta = 90°$ and the hyperbola is rectangular. Note that the ellipse has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $c^2 = a^2 - b^2$. If we use $PO^2 + PF_1 \cdot PF_2 = k$ to define the same ellipse, then $k = a^2 + b^2 = a^2(2 - e^2)$. Again, if we let $e = sech\, \theta$ and $c = ae$, then $k = \frac{c^2}{e^2}(2 - e^2) = c^2(2\cosh^2\theta - 1) = c^2 \cosh 2\theta$. Here the geometrical meaning of θ is less obvious. The value of k can only be positive because $k = c^2 \cosh 2\theta$ suggests that $k > c^2$.

In conclusion, we have seen that $PO^2 \pm PF_1 \cdot PF_2 = k$ defines a central conic. We emphasize, however, that this approach cannot be extended to define a parabola since a parabola has only one focus F_1. If a second focus F_2 is assumed to be at infinity, then the line segment F_1F_2 has infinite length, and the present approach will not work.

Figure 1