EXPANSIONS OF BAIRE SPACES

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Let (X, T) be a topological space. Following Hewitt [3], if T' is a topology on X such that $T \subset T'$ we call (X, T') an expansion of (X, T). Several other authors [1, 2, 3, 4, 6, 7, 8] have subsequently studied the preservation of topological properties under expansions. In this note we consider the preservation of Baire spaces under expansions. A Baire space is a topological space in which every nonempty open set is of second category. Equivalently, a space is Baire if and only if none of its nonempty open sets is the union of countably many nowhere dense sets; this is true if and only if the intersection of every sequence of dense open sets is dense.

We show that an expansion of a Baire space need not be Baire, and that the supremum of a collection of Baire topologies need not be a Baire topology. We give some sufficient conditions under which an expansion of a Baire space is Baire. It would be interesting to have necessary conditions under which stronger topologies are Baire when the original topology is Baire.

We refer the reader to [5] for definitions of the basic topological notions used herein. In the sequel, we use the symbol A^c to denote the complement of a set A. We begin with an example.

Example 1. The topological expansion of a dense–in–itself Baire space to a dense–in–itself space need not be Baire.

Construction. Let (X, T_1) be a dense–in–itself Baire space with a first category dense subset A, e.g., let $X = [0,1]$ with the usual topology, and let A be the subspace of rational numbers in $[0,1]$. Let T_2 be the topology generated by $T_1 \cup \{A\}$. Each member of T_2 is of the form $(A \cap G_1) \cup G_2$ for some $G_1,\ G_2 \in\ T_1$. Since A is of the first category in (X, T_1),

$$A = \bigcup_{n=1}^{\infty} C_n$$

where each C_n is nowhere dense. Consider a fixed C_n. To see that C_n is nowhere dense in (X, T_2), it suffices to observe that if $(A \cap G_1) \cup G_2$ is a T_2–open set with $A \cap G_1 \neq \emptyset$, there exists a nonempty T_1–open subset G_3 of G_1 such that $G_3 \cap C_n = \emptyset$. It follows that the T_2–open set A is of the first category in (X, T_2).

Hence, (X,T_2) is not Baire. That (X,T_2) is dense–in–itself is evident.

We now give some sufficient conditions for expansions of Baire topologies to be Baire.

Proposition 1. Suppose (X,T_1) is a space, $A \subset X$, and T_2 is the topology on X generated by $T_1 \cup \{A, A^c\}$. Then each nowhere dense subset of (X,T_2) is also nowhere dense in (X,T_1). Thus if (X,T_1) is a Baire space, if A is second category in each T_1–open set which it meets, and if A^c is second category in each T_1–open set which it meets, then (X,T_2) is a Baire space.

Proof. Suppose E is nowhere dense in (X,T_2). Let G be a nonempty T_1–open set such that $G \cap E \neq \emptyset$. To show that E is nowhere dense in (X,T_1), it suffices to show that G contains a nonempty T_1–open set H such that $H \cap E = \emptyset$.

Case 1. $A \cap G \cap E \neq \emptyset$. Since E is nowhere dense in (X,T_2), there exist G_1', $G_2' \in T_1$ such that $\emptyset \neq (A \cap G_1') \cup G_2' \subset A \cap G$ and $[(A \cap G_1') \cup G_2'] \cap E = \emptyset$. If $G_2' \neq \emptyset$, take $H = G_2'$. Otherwise, $\emptyset \neq A \cap G_1' \subset A \cap G$ and $A \cap G_1' \cap E = \emptyset$. Let $G_1 = G_1' \cap G \subset G$.

Then

$$\emptyset \neq A \cap G_1' \cap G = A \cap G_1 \subset A \cap G.$$

If $A^c \cap G_1 \cap E = \emptyset$, then $G_1 \cap E = \emptyset$ and we take $H = G_1$. If $A^c \cap G_1 \cap E \neq \emptyset$, since E is nowhere dense in (X, T_2), there exist $G_2,\ G_3 \in T_1$ such that $\emptyset \neq (A^c \cap G_2) \cup G_3 \subset A^c \cap G_1$ and $[(A^c \cap G_2) \cup G_3] \cap E = \emptyset$. Again, if $G_3 \neq \emptyset$, set $H = G_3$. Otherwise, $\emptyset \neq A^c \cap G_1 \cap G_2$, and $A^c \cap G_1 \cap G_2 \cap E = \emptyset$. Then $G_1 \cap G_2$ may be taken as the required subset H of G. For evidently, $G_1 \cap G_2$ is a nonempty T_1–open subset of G. Moreover, $G_1 \cap G_2 \cap E = \emptyset$; otherwise $G_1 \cap G_2$ would contain points of E but no point of A^c, and we would have

$$\emptyset \neq G_1 \cap G_2 \cap E \cap A \subset A \cap G_1 \cap E = \emptyset.$$

<u>Case 2</u>. $A \cap G \cap E = \emptyset$. Then $A^c \cap G \cap E \neq \emptyset$, and the existence of the required subset H of G follows by symmetry. The remainder of the proposition is easily established.

<u>Proposition 2</u>. Suppose (X, T_1) is a space and A is a subset of X such that A^c is of the first category. Let T_2 be the topology on X generated by $T_1 \cup \{A\}$. Then every first category subset of (X, T_2)

is first category in (X, T_1). Thus if (X, T_1) is a Baire space, and if A is residual, then (X, T_2) is a Baire space.

Proof. We will show that if G is T_2–open and dense, then G^c must be category 1 in (X, T_1). This will imply that every subset of X which is nowhere dense in (X, T_2), and, consequently, every subset of X which is category 1 in (X, T_2), is category 1 in (X, T_1). For if E is nowhere dense in (X, T_2), then $X - \overline{E}$ is T_2–open and dense. Hence $\overline{E}$ is category 1 in (X, T_1), and, consequently, so is E.

Assume, therefore, that G is T_2–open and dense. If G is T_1–open, then G^c is nowhere dense and hence category 1 in (X, T_1). Otherwise, there exist $G_1, G_2 \in T_1$ such that $G = (A \cap G_1) \cup G_2$ with $A \cap G_1 \neq \emptyset$. Let H be the interior of G_2^c in (X, T_1) furnished with the relativized T_1 topology. Now $G_1 \cap H$ is dense in H. Otherwise, there would exist a nonempty open subset G_3 of H such that $G_3 \cap G_1 \cap H = \emptyset$. Hence $G_3 \cap [(A \cap G_1) \cup G_2] = G_3 \cap G = \emptyset$, contrary to the assumption that G is T_2–dense. It follows that $H - G_1$, the boundary of $G_1 \cap H$ in H, is nowhere dense, and hence of category 1 in H. Hence $H - G_1$ is category 1 in (X, T_1). Since bdry G_2

(the boundary of G_2 in (X, T_1)) and $H - A$ are both category 1 subsets of (X, T_1), and since $G^c = (H - A) \cup (H - G_1) \cup \text{bdry } G_2$, it follows that G^c is category 1 in (X, T_1). The remainder of the proposition follows easily.

The following example shows that the supremum of an increasing sequence of Baire topologies need not be Baire.

Example 2. There exists a dense–in–itself Baire space (X, T_0) and a sequence $T_0, T_1, T_2, \cdots$ of topologies on X such that for each $i, T_i, \subset T_{i+1}$ and (X, T_i) is Baire, but (X, T) is not Baire if T is the supremum of the sequence $\{T_n\}$.

Construction. Let X be the space of real numbers with the usual topology. By standard methods we decompose X into a sequence $A_0, A_1, A_2 \cdots$ of pairwise disjoint sets such that for each nonempty open set G, and for each i, $A_i \cap G$ is nonempty and second category. We divide $\{A_0, A_1, A_2, \cdots\}$ into classes $C_0, C_1, C_2, \cdots$ so that $C_k = \{B(0, k), B(1, k), \cdots, B(2^k - 1, k)\}$ where

$$B(i, k) = \bigcup_{j=0}^{\infty} A_{i+j}(2^k),\ 0 \leq i \leq 2^k - 1\ .$$

For each k,

$$X = \bigcup C_k = \bigcup_{i=0}^{2^k-1} B(i,k),$$

$$B(i,k)^c = \bigcup_{\ell=1,\ell\neq i}^{2^k-1} B(\ell,k)\ ,$$

and

$$B(i,k) \cap B(\ell,k) = \emptyset \text{ if } i \neq \ell\ .$$

Let T_0 be the usual topology for the space X of real numbers. (X, T_0) is a Baire space. Having constructed Baire topologies $T_0 \subset T_1 \subset \cdots \subset T_k$ on X, let T_{k+1} be the expansion of T_k generated by

$$T_k \cup \big(\bigcup_{i=1}^{2^k-1} \{B(i,k), B(i,k)^c\}\big)\ .$$

By Proposition 1, each of the spaces $(X,T_0),(X,T_1),(X,T_2),\cdots$ is a Baire space and $T_i \subset T_{i+1}$ for each i. Let T be the supremum of $T_0, T_1, T_2, \cdots$. We assert that (X,T) is not Baire. To prove this we will show that A_n is nowhere dense in (X,T) for each n. So consider a fixed set A_n. Let H be a nonempty T–open set. We will show that H contains a nonempty T–open set which misses A_n. Note first that there is a nonempty T_0–open set G and a corresponding basic T–open set $G \cap B(i,k)$ such that $\emptyset \neq G \cap B(i,k) \subset H$ for some nonnegative integers i and k. If $n \neq i + j(2^k)$ for some nonnegative j, then $A_n \cap B(i,k) = \emptyset$, and consequently, $G \cap B(i,k) \cap A_n = \emptyset$. If $n = i + j(2^k)$ for some j, then

$$B(i + (j+1)2^k, k + (j+1)) \subset B(i,k)$$

and

$$B(i + (j+1)2^k, k + (j+1)) \cap A_n = \emptyset.$$

Hence

$$G \cap B(i + (j+1)2^k, k + (j+1)) \neq \emptyset,$$

and

$$G \cap B(i + (j+1)2^k, k + (j+1)) \cap A_n = \emptyset.$$

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