CONVERGENCE SEMIGROUPS

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In [1], B. Pearson defined convergence spaces and investigated some of their structures. It was pointed out that convergence spaces are much weaker in structure than topological spaces. The purpose of this paper is to define and investigate convergence semigroups along the same line of thought as in chapter 1 of [2]. Although weaker in structure, most of the results discussed in [2] or [3] on topological semigroups are found to be true on convergence semigroups.

Definition. The set D is directed by the relation $\geq$ if (1) for each m in D, $m \geq m$, (2) for each m, n, and p in D, if $p \geq n$ and $n \geq m$, then $p \geq m$, and (3) for each m and n in D, there exists p in D such that $p \geq m$ and $p \geq n$. If D is directed by the relation $\geq$ and $m \in D$, then $\underline{\{n \in D \mid n \geq m\}}$ will be denoted by mD. The domain and range of a relation f will be denoted by $D(f)$ and $R(f)$ respectively. A net is a map s such that $D(s)$ is a directed set. If

$m \in D(s)$, then $s(m)$ will sometimes be denoted by s_m. If X is a set and s is a net with domain D, then s is in X if $R(s) \subseteq X$, s is eventually in X if for some m in D, $s(mD) \subseteq X$. A net t is a subnet of s, denoted by $t \leq s$, if for each $m \in D$, t is eventually in $s(mD)$.

Definition. A convergence structure on the set X is a class C of ordered pairs such that (1) if $(s,x) \in C$, then s is a net in X and $x \in X$, and (2) if $(s,x) \in C$ and t is a net in X such that $t \leq s$, then $(t,x) \in C$. If C is a convergence structure on X, then (X,C) or just X is called a convergence space and the statement that s converges to x, denoted by $s \to x$ or $lim\ s = x$, means $(s,x) \in C$.

Definition. Suppose X is a convergence space and $Y \subseteq X$. For each point $x \in Y$ and each net s in Y let $s \to x$ in Y if and only if $s \to x$ in X. Then Y is called a subspace of X.

In the following definitions and theorems, X is a convergence space.

Definition. X is said to have the constant convergence property if for each x in X and net s in $\{x\}$, $s \to x$.

Theorem 1. If X has the constant convergence property and Y is a subspace of X, then Y has the constant convergence property.

Proof. Suppose $y \in Y$ and s is a net in $\{y\}$. Then s is a net in Y. Since $Y \subseteq X$ and X has the constant convergence property, $s \to y$ in Y. Since Y is a subspace of X, $s \to y$ in Y and hence Y has the constant convergence property.

Definition. If $M \subseteq X$, then $\overline{M}$ is the set of all points x such that some net in M converges to x.

Theorem 2. If X has the constant convergence property and $A \subseteq X$, then $A \subseteq \overline{A}$.

Proof. Let $a \in A$. Let s be a net such that $R(s) = \{a\}$. Since X has the constant convergence property, it follows from Theorem 1 that $s \to a$ in A and hence $a \in \overline{A}$. Therefore $A \subseteq \overline{A}$.

Theorem 3. If f is a map from X to Y, s is a net in X, and $t \leq f \circ s$, then there is a subnet u of s such that $f \circ u \leq t$.

Proof. Let $D = D(s)$ and $F = D(t)$. Then $D(f \circ s) = D$. Let $G = \{(m, n) \in D \times F | t_n = f(s_m)\}$ with the cross product order. It is easy to see that G is a directed set. Let u be the net with domain G defined by $u(m, n) = s_m$. Let $p \in D$. Since $t \leq f \circ s$, there exists $q \in F$ such that $t(qF) \subseteq s(pD)$. Let $(m, n) \geq (p, q)$.

Then $m \geq p$ and $u(m,n) = s_m$. Thus $u \leq s$. Let $k \in F$. Since $t \leq f \circ s$, there exists $q \geq k$ such that $t_q = f(u(p,q))$ for some p in D. Let $(m,n) \geq (p,q)$. Then $n \geq k$ and $f(u(m,n)) = t_n$. Therefore $f \circ u \leq t$.

Definition. Let f be a map from X to Y. Then the statement that f is strongly continuous means that if $x \in X$ and s is a net in X such that $s \to x$, then $f \circ s \to f(x)$. And, f is continuous means that if $x \in X$ and s is a net in X such that $s \to x$, then some subnet of $f \circ s$ converges to $f(x)$.

Definition. The statement that X is pseudotopological at x means that if s is a net in X such that each universal subnet of s converges to x, then $s \to x$.

Theorem 4. If f is a continuous map from X to Y and Y is pseudotopological, then f is strongly continuous.

Proof. Let $s \to x$, and let t be a universal subnet of $f \circ s$. By Theorem 3, there is a net u in X such that $u \leq s$ and $f \circ u \leq t$. Since $u \to x$ and f is continuous, there exists a subnet v of $f \circ s$ such that $v \to f(x)$. Since t is universal, $t \leq v$ and hence $t \to f(x)$. Since every universal subnet of $f \circ s$ converges to $f(x)$ and Y is pseudotopological, $f \circ s \to f(x)$. Therefore f is strongly continuous.

Definition. X is Hausdorff if no net in X converges to two distinct points of X and X is compact if each net in X has a convergent subnet.

Definition. A convergence semigroup is a semigroup X with a convergence structure and the multiplication defined on X is strongly continuous, which means that if s and t are nets in X with the same domain such that $s \to x$ and $t \to y$, then $st \to xy$.

Definition. Let A be a nonempty subset of a convergence semigroup X. Then A is called a convergence subsemigroup of X if $aA \subseteq A$ for each $a \in A$ and A is called a convergence subgroup of X if $aA = Aa = A$ for each $a \in A$.

In the following theorems and definitions, X is a convergence semigroup.

Theorem 5. Let X be a convergence semigroup with more than one element. Then X contains a convergence subsemigroup A such that $A \neq X$.

Proof. Suppose $A = X$ for every convergence subsemigroup A of X. Then $xX = X = Xx$ for every x in X and therefore X

is a group. Let e be the identity of X. It is clear that $\{e\}$ is a subsemigroup of X. According to Theorem 1, $\{e\}$ is a subspace of X and hence $\{e\}$ is a convergence subsemigroup of X. By the supposition, $\{e\} = X$. But this contradicts the assumption that X has more than one element.

Theorem 6. If s and t are nets in X with domains D and E respectively, then there exists nets u and v in X such that $D(u) = D(v)$, $u \leq s$, and $v \leq t$.

Proof. Let $D = D(s)$ and let $F = D(t)$. Let $G = D \times F$ with the cross product order. Then it is clear that G is a directed set. Let u and v be nets with domain G such that $u(m, n) = s_m$ and $v(m, n) = t_n$. Let $p \in D$. Then for each $m \in F$ and $q \geq p$, $u(m, q) \in s(pD)$. Therefore $u \leq s$. Similarly, $v \leq t$.

The next theorem follows immediately from Theorem 6 and the definition of convergence semigroup.

Theroem 7. If s and t are nets in X such that $s \to x$ and $t \to y$, then there exist nets u and v in X such that $u \leq s$, $v \leq t$, and $uv \to xy$.

Theorem 8. If A is a convergence subsemigroup of X, then $\overline{A}$ is a convergence subsemigroup of X.

Proof. Let x, y be elements of $\overline{A}$. Then there exist nets s and t in A such that $s \to x$ and $t \to y$. According to Theorem 7, there exist nets u and v in X such that $u \leq s$, $v \leq t$ and $uv \to xy$. Therefore $xy \in \overline{A}$ and hence $\overline{A}$ is a convergence subsemigroup of X.

The proof of the next theorem is the same as the proof of Theorem 1.1.3 in [2].

Theorem 9. Each subgroup of a convergence semigroup X is contained in a (unique) maximal subgroup, and no two maximal subgroups of X intersect.

Definition. M is said to be closed in X if $\overline{M} \subseteq M$.

Theorem 10. Let X be a compact, Hausdorff convergence semigroup with the constant convergence property. If A is a subgroup of X and $a \in \overline{A}$, then $b \in a\overline{A}$ ($b \in \overline{A}a$) for every b in $\overline{A}$.

Proof. Let $a \in \overline{A}$. Then there exists a net s in A with domain D such that $s \to a$. Let t be the net with domain D such that for each $i \in D$, $t_i = s_i^{-1}$. Since X is compact, there exists a subnet t' of t such that $t' \to x$ for some x in $\overline{A}$. By Theorem 7, there exist subnets u and v of s and t' respectively such that $uv \to ax$.

Let e be the identity of A. Then uv is a net in $\{e\}$. Since X is Hausdorff and X has the constant convergence property, it follows that $ax = e$. Similarly, $xa = e$ and hence $x = a^{-1}$. Let $b \in \overline{A}$. Then there exists a net w in A such that $w \to b$. By Theorem 7, there exist subnets v' and w' of v and w respectively such that $v'w' \to a^{-1}b$ and $a^{-1}b \in \overline{A}$. Since $b = aa^{-1}b$, $b \in a\overline{A}$. The proof of the statement $b \in \overline{A}a$ is somewhat similar.

<u>Theorem 11</u>. If X is a Hausdorff, compact convergence semigroup, then each maximal subgroup of X is closed.

<u>Proof</u>. Let A be a maximal subgroup of X and let x, y be elements of $\overline{A}$. According to Theorem 8, $xy \in \overline{A}$. Thus $x\overline{A} \subseteq \overline{A}$ and $\overline{A}y \subseteq \overline{A}$. Let $z \in \overline{A}$. By Theorem 10, $\overline{A} \subseteq z\overline{A}$ and $\overline{A} \subseteq \overline{A}z$. Therefore $z\overline{A} = \overline{A} = \overline{A}z$ for every $z \in \overline{A}$ and hence $\overline{A}$ is a subgroup of X containing A. By the maximality of A, $\overline{A} = A$.

<u>Definition</u>. A <u>convergence group</u> is a group with a convergence structure and the multiplication defined on X is strongly continuous.

<u>Definition</u>. An element e of a convergence semigroup X is called an <u>idempotent</u> if $e^2 = e$. We shall denote by E the set of idempotents

in X.

Theorem 12. If X is a Hausdorff convergence semigroup, then the set E of all idempotents of X is closed.

Proof. If $E = \emptyset$, then the theroem is trivial. Let $x \in \overline{E}$. Then there is a net s in E such that $s \to x$. Since $lim\ ss = xx = x^2$ and $lim\ ss = lim\ s = x$, $x^2 = x$ and so $x \in E$. Therefore E is closed.

The proof of the following theorem is the same as the proof of Theorem 1.8 in [3].

Theorem 13. If X is a compact convergence semigroup, then X contains at least one idempotent.

Theorem 14. Let X be a Hausdorff, compact, pseudotopological convergence semigroup with the constant convergence property. If X is an abstract group, then X is a convergence group.

Proof. Let m be the map from $X \times X$ to X and n be the map from X to X such that if x, y are elements of X, then $m(x, y) = xy$ and $n(x) = x^{-1}$. Since X is a convergence semigroup, m is strongly continuous. Let $a \in X$ and let s be a net with domain D such that $s \to a$. Then it is obvious that $D(n \circ s) = D$ and for each $k \in D$, $n(s_k) = s_k^{-1}$. Since X is compact, there exists a subnet t of $n \circ s$

such that $t \to b$ for some b in X. By Theorem 7, there exist nets u and v such that $u \leq s$ and $v \leq t$ and $uv \to ab$. Let e be the identity of X. Since uv is a net in $\{e\}$, X is Hausdorff, and X has the constant convergence property, $ab = e$. Similarly, $ba = e$ and hence $b = a^{-1}$. Therefore n is continuous. It follows from Theorem 4 that n is strongly continuous and thus X is a convergence group.

Theorem 15. If X is a convergence semigroup with the constant convergence property and X' is a compact convergence subsemigroup of X, then xX' is a compact convergence subsemigroup of X for every x in X'.

Proof. Let $x \in X'$ and let xa and xb be elements of xX'. Then $axb \in X'$ and hence $xaxb \in xX'$. It follows that xX' is a convergence subsemigroup of X. Let s be a net in xX' with domain D. Then for each $i \in D$, $s_i = xa_i$ for some a_i in X'. Let t be the net with domain D such that for each $i \in D$, $t_i = a_i$. Then t is a net in X'. Since X' is compact, there exists a subnet t' of t with domain F such that $t' \to y$ for some y in X'. Let u be the net with domain F such that $R(u) = \{x\}$. Let $s' = s|F$. Since X has the constant convergence property, $u \to x$. Since X' is a convergence subsemigroup, $lim\ s' = lim\ ut = xy$. Therefore xX' is a compact convergence subsemigroup of X.

Theorem 16. Let X be a compact, convergence semigroup with the constant convergence property and X' be a compact convergence subsemigroup of X. If X is an abstract group, then X' is a subgroup.

Proof. According to Theorem 13, X' contains an idempotent e which is the identity of X. Let $x \in X'$. By Theorem 15, xX' is a compact convergence subsemigroup of X. Applying Theorem 13 to xX', there is an idempotent e' in xX'. Since $ee' = e'e'$, $e = e'$ and hence $e \in xX'$. Therefore $X' = eX' \subseteq xX'$. Since X' is a subsemigroup, $xX' \subseteq X'$ and thus $xX' = X'$. Similarly, $X'x = X'$. Therefore X' is a subgroup.

Definition. Let X be a semigroup. The statement that X satisfies the $\underline{\text{two} - \text{sided cancellation law}}$ means that for all a, b, and c in X, $ac = bc$ implies $a = b$ and $ca = cb$ implies $a = b$.

Theorem 17. If X is a compact, pseudotopological convergence semigroup, and X satisfies the constant convergence property and the two–sided cancellation law, then X is a convergence semigroup.

<u>Proof</u>. Let $x \in X$. It is clear that $xX \subseteq X$ and $Xx \subseteq X$. According to Theorem 16, X contains at least one idempotent. Let e and e' be idempotents in X. Since $eee' = ee'$ and $ee'e' = ee'$, it follows from the two–sided cancellation law that $ee' = e'$ and hence $e = e'$. Therefore there is only one idempotent, denoted by 1, in X. Now $1^2x = 1x$. It follows from the two–sided cancellation law that $1x = x$. Similarly, $x1 = x$. Therefore 1 is the identity of X. Since $x = x1 = 1x$, $X \subseteq xX$ and $X \subseteq Xx$ and hence $xX = Xx = X$. According to Theorem 14, X is a convergence group.

References

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