HOW MATHEMATICS GROWS

Tran Van Thuong and Barry E. Brown

Missouri Southern State College

Mathematics grows on all levels. One doesn't need a Ph.D. — as is the case in other sciences — to make interesting and significant contributions to the field. Even at the lower levels, students can be challenged to initiate research projects, and to develop novel solutions to problems of their own devising.

In this paper, we explore some of the procedures by which mathematics grows. We also offer several original examples of varying difficulty that demonstrate these procedures. Our hope is that instructors can use these examples, and the methodology that underlies them, to motivate students to think of mathematics as an enterprise of which they (the students) are a part. If students (and instructors) feel that there are still many problems on the lowest levels of mathematics in need of formulation and solution, courses in mathematics can become more stimulating and alive.

1. Observation and Explanation

Typically, the starting point for the mathematician is observation. After observing some unusual — or not so unusual — event, he/she will seek to explain what he/she has observed mathematically.

As an example, suppose our mathematician visits a Las Vegas Casino, and observes the following four–card magic trick: the dealer presents a brand new fifty–two card deck and asks the audience to cut the cards as many times as they please, once at a time (here, cutting once at a time means cutting the top of the deck, putting the cut cards to the right, and placing the bottom cards on top of them). The dealer then deals four cards to thirteen different people, and "magically" each person receives four cards of the same kind — i.e., four aces, four kings, four deuces, etc.

His curiosity piqued, our mathematician might naturally seek to explain this "phenomenon". He can do so in a non–technical and understandable manner using a standard technique of geometric analysis.

In a brand new deck, the cards are arranged in the following order: on top is the ace of hearts, then the deuce of hearts, the three of hearts, $\cdots$, the ten of hearts, jack of hearts, queen of hearts, king of hearts; then the ace of spades, deuce of spades, and so on. Suppose then that we represent the deck by means of a circle divided into fifty–two equal parts, denoted respectively by $1H$, $2H$, $3H, \cdots$, $11H$, $12H$, $13H$, $1S$, $2S$, and so on (note that "$11H$" stands for the jack of hearts, "$12H$" for the queen of hearts, and "$13H$" for the king of hearts) and that we sequence the cards counterclockwise, letting the "top" of the circle represent the top of the deck.

Figure 1

Since there are thirteen cards in each suit, each 90 degree segment of the circle has thirteen parts, and each card in a suit is separated from its counterpart in the nearest suit by 90 degrees.

And if the cards are dealt to thirteen different people, each person will then receive the cards that are separated by 90 degrees — the first person will receive $1H$, $1S$, $1D$, and $1C$, the second person $2H$, $2S$, $2D$, and $2C$, and so on.

Now cutting the deck amounts simply to rotating the circle in a clockwise direction. Since each card is separated from its successor by 90/13 degrees, a cut of three cards amounts to rotating the deck 3 x (90/13) degrees, a cut of 10 cards by 10 x (90/13) degrees, and so on. In no case, though, is the order of the cards disturbed. All that changes is which card is on top. However, so long as thirteen different people are dealt cards, each person will still receive cards at 90 degrees to each other — namely, the same cards in the different suits. If three cards are cut, for example, $4H$ is then the top card on the deck and $3H$ the bottom card. The first person will then receive four fours and the last person four threes. In general, if we cut the new deck n times, the first time with n_1 cards, and the nth time with n_n cards, our action is equivalent to a clockwise rotation of the circle by an angle of $(n_1 + n_2 + \cdots + n_n)x$, where $x = 90/13$ degrees. However, again it is obvious that no matter how many times the deck is cut, the sequence of the cards is not changed, and hence the "trick" will still "work".

2. Extending Existing Knowledge in Mathematics

Explaining observed phenomena is one method by which mathematics advances. Another is by extending existing knowledge. For example, after observing and explaining the four card magic trick, the mathematician might be tempted to ask:

(1) Is it necessary to use a deck arranged as brand new decks are arranged, or will the trick work with alternative arrangements of the cards?

(2) Can we use algebraic methods to predict that the trick will still work?

and

(3) Can we arrange the deck in some other specific order so that everyone receives two different pairs?

In fact, the answer to each of these questions is "yes" [1]. Using our circle diagram, for example, we can see that the trick could be made to work if we eliminated the picture cards from the deck and dealt out the remaining cards to ten different people. In this case, each 90 degree segment of the circle would be divided into ten equal parts. We could insure that each person received two pairs if we arranged the standard fifty–two card deck in, for example, the following manner:

Figure 2

In fact, the reader ought to be able to work out the many variations on the "standard" trick from which this discussion began, and teachers might want to assign the development of such variations as exercises for their students.

Another (slightly more complex) example of extending existing knowledge can be drawn from the study of quadratic equations. College algebra shows us that the solutions for a quadratic equation $ax^2+bx+c=0$ are $(-b\pm\sqrt{b^2-4ac})/2a(a\neq 0)$. If the discriminant $D=(b^2-4ac)\geq 0$, then we have two real solutions or a double real solution. College algebra also shows us how to see the graphical representation of two real solutions or one double real solution by looking at the intersection of the points of the graph of $f(x)=ax^2+bx+c$ and the x–axis (Figure 3):

Figure 3

Students learning to solve quadratic equations should not, however, be satisfied with learning only this standard part of the curriculum. They should, rather, seek to extend their knowledge by asking "what happens if the discriminant $(D=b^2-4ac)<0$?" Of course, the graph of $f(x)=ax^2+bx+c$ does not intersect the

x–axis when $(D = b^2 - 4ac) < 0$ (Figure 4).

Figure 4

Instead, if $D = (b^2 - 4ac) < 0$, we have two conjugate complex roots, namely, $x = (-b \pm \sqrt{4ac - b^2}\ i)/2a$. The question then becomes "how can the two conjugate complex roots of the equation be graphically represented?"

To see how this question can be answered, it will be useful to consider a particular case and then generalize. Let us then consider the particular equation $x^2 - 2x + 5 = 0$. $x_1 = 1 + 2i$, and $x_2 = 1 - 2i$ are the two conjugate complex solutions of this equation. To graphically represent these solutions, we then, first, construct a three–dimensional coordinate–system xyz, in which the plane spanned by ox and oy is the complex plane, and ox represents the real part of the complex numbers. Observing that $f(x) = x^2 - 2x + 5 = (x - 1)^2 + 4$, we next graph the function $F_1(x) = x^2 + 4$ in the coordinate–system $o'XY$. The graph of this function is a parabola, concave up, with its vertex at (0,4).

Figure 5

The next step is to rotate F_1, an angle of 90 degrees counterclockwise about the Y–axis. The result is F_2 in the coodinate system $o'XY$. Note that in this coordinate–system, the graph of F_2 is exactly the graph of the function $F_2(x) = x^2 + 4$. Now we flip the graph F_2 over on the straight line $Y = 4$ in $o'XY$, and obtain the graph F_3, which is the graph of the function $F_3(x) = -x^2 + 4$ in the coordinate system $o'XY$. It is easy to see that the intersection points of F_3 and the X–axis are the two points $A(2, 0)$, and $B(-2, 0)$. In fact, A and B represent the two complex numbers $1 + 2i$ and $1 - 2i$, respectively, in the complex plan oxy. The two complex conjugate roots of the equation $x^2 - 2x + 5$ are thus represented by the two intersection points A and B of the graph F_3

and the X–axis. The reader should now see that it is not difficult to generalize this technique for any two complex solutions of $ax^2 + bx + c = 0$.

This example was, again, given to illustrate the concept of extending existing knowledge in mathematics. In this case, we sought to extend our knowledge of how to represent real solutions of the quadratic equation $ax^2 + bx + c = 0$ to the graphical representation of two conjugate complex solutions of $ax^2 + bx + c = 0$. At this point, we should, of course, wonder about how we might further extend our knowledge; i.e., we should wonder whether our solution can be further extended to the geometrical representation of complex roots of any polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$. As yet, this question remains to be explored.

3. Looking for Applications

Finally, another method by which mathematics advances is by attempting to apply theorems to new domains — i.e., by looking for applications of existing knowledge. As an illustration, let's consider yet another example. The following theorem is a consequence of the distributive laws governing real numbers:

Theorem: For any real numbers x and y, where $5 < x \leq 10$, and $5 < y \leq 10$, $xy = 10((x-5)+(y-5)) + ((10-x)(10-y))$.

Most mathematical theorems make sense if one can see their applications, and this theorem leads to an interesting application concerning multiplication of two real numbers from six to ten.

First, we number the finger of the two hands as follows:

Figure 6

That is to say, with palms facing inward, we number the thumb of each hand 10, the index finger 9, the middle finger 8, the ring finger 7, and the little finger 6. We are now prepared to use our fingers (and the theorem mentioned above) to multiply numbers between 6 and 10.

Suppose, for example, that we want to compute the product of 6 and 9. Keeping the palms facing inward, touch the index finger (9) of the right hand to the little finger (6) of the left hand,

Figure 7

count the number of fingers below and including the touching fingers (there are five), and multiply by 10 (the result is of course 50). Next, multiply the number of fingers of the left hand about (but not including) the touching fingers (there is one) by the number of fingers of the right hand above (but not including) the touching fingers (there are four), and add the product ($4x1 = 4$) to the previous result. The final answer is of course 54.

It is easy to verify that this technique works for products of all numbers between six and ten. To multiply six by six, for example, we touch the little fingers of both hands, noting that there are two fingers below and including those touching, and four above on each hand. Then we multiply two by ten, and four by four, and add the products to arrive at our answer: 36. To multiply six by seven, we touch the ring finger of the left hand to the little finger of the right hand, noting three fingers below those touching, three above on the right hand, and four above on the left hand. We then multiply three by ten, and three by four, and add the products to arrive at our answer: 42.

Having mastered this application of an already known theorem, the mathematician should now look for ways to extend what he has discovered. He might wonder, for example, if a more general method for multiplying by hand can be devised that works for any real numbers. One fairly simple extension involving multiplication of numbers between 10 and 15 can, for example, be devised using the following theorem:

Theorem: For any real numbers x and y, where $5 < x \leq 10$ and $10 < y \leq 15$, $\quad xy = 10(x - 5 + y - 10)(15 - y) + 5x$.

In order to make use of this theorem, we number the fingers of the left hand as before, but number the fingers of the right hand as illustrated below:

Figure 8

The technique for computing products is identical to the previous technique except that now we must add $5x$ (where x is our initial multiple) to our result. For example, suppose we want to multiply 7 by 12. First we touch the ring fingers of both hands together, noting that there are two fingers below and including those touching, and three above, on each hand. Next we multiply 10 by 4 and 3 by 3, and add the results (49). Finally, we multiply the initial multiple (7) by 5, and add this product (35) to the previous sum. Our answer is then $40 + 9 + 35 = 84$.

Mathematics grows from the attempt to answer the sorts of questions raised in this paper: how can we explain what we observe? how can we extend what we have learned? and how can we apply what we now know? As should be obvious, the process has no fixed limit; there is no point at which mathematics will stop growing, no completion, even ideally, to the mathematical enterprise. It make sense — at least according to some — to talk about an "ideally completed" physics or psychology or biology, about an ideal physical or psychological or biological theory that explains and predicts all of the facts falling under the discipline. But no such ideal applies to mathematics. As long as mathematicians continue to ask and to answer questions, the discipline will continue to grow.

References

1. T. V. Thuong, "Four Card Magic," Submitted to *Illinois Journal of Mathematics* (1989).