ON THE LOWER NEAR FRATTINI SUBGROUPS AND NEARLY SPLITTING GROUPS

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In this article, first we briefly reintroduce the Frattini subgroup $\Phi(G)$, and the lower near Frattini subgroup $\lambda(G)$, of a group G. Second, we prove a lemma, due to C. Y. Tang, concerning the Frattini subgroups and splitting groups. Also, we prove the exact analog of Tang's lemma for the lower near Frattini subgroups and nearly splitting groups. Finally, we propose a question for the reader.

The Frattini Subgroup. The Frattini subgroup, $\Phi(G)$ of a group G, was introduced into the group theory first in 1885 by the Italian mathematician Giovanni Frattini (1852–1925). $\Phi(G)$ is the intersection of all maximal proper subgroups of G (if there are no maximal proper subgroups of G, then $\Phi(G) = G$). It is easy to see that $\Phi(G)$ is the set of all nongenerators of G. (Recall: An element g of a group G is a nongenerator of G if for every subset S

of G such that $< S, g > = G$, then $< S > = G$). For more results concerning $\Phi(G)$ see [5].

The Lower Near Frattini Subgroup. As it appears from the name, the lower near Frattini subgroup of a group G is a close analog of the Frattini subgroup of G in many ways. The lower near Frattini subgroup of a group was first published in 1969 by J. B. Riles. All definitions in this article (except definitions 4 and 5) are due to him [3].

Definition 1. An element g of a group G is a near generator of G if there is a subset S of G such that $|G : < S >|$ is infinite, but $|G : < g, S >|$ is finite.

Definition 2. An element g of a group G is a non – near generator of G if $S \subseteq G$ and $|G : < g, S >|$ is finite imply that $|G : < S >|$ is finite.

Observe that if G is a finite group, then every element of G is a non–near generator of G.

The idea for Propositions 1 and 2 came from [3].

Proposition 1. The set of all non–near generators of a group G is a subgroup of G.

Proof. Since the identity of G is a non–near generator of G, the set of non–near generators of G is not empty. Therefore, it is enough to show that, if x and y are any two non–near generators of G, then so is xy^{-1}. Let $S \subseteq G$ be such that $|G : < xy^{-1}, S >|$ is finite. (In what follows, $\leq$ will denote the subgroup relation.) From $< xy^{-1}, S > \leq < x, y, S >$ it follows that $|G : < x, y, S >|$ is finite. Thus, $|G : < S >|$ is finite. Hence, xy^{-1} is a non–near generator of G. This completes the proof.

Defintion 3. The set of all non–near generators of a group G is called the lower near Frattini subgroup of G, denoted by $\lambda(G)$.

The Main Result. In 1972, C. Y. Tang published the following lemma (without proof) concerning the Frattini subgroups and splitting groups [6, Lemma 2.6, p. 570]. We restate this lemma for the lower near Frattini subgroups and nearly splitting groups (as a theorem), and we prove both the lemma and the theorem. But first we need the following three well–known definitions.

Definition 4. If H is a subgroup of a group G, then

$$K(G, H) = \bigcap_{g \in G} g^{-1} H g$$

is called the core of H in G. $K(G, H)$ is the unique largest normal subgroup of G contained in H.

Definition 5. Let H be a normal subgroup of a group G. We say that G splits over H if there is a subgroup K of G such that $G = HK$ and $H \cap K = 1$. Any such subgroup K is said to be a complement to H in G.

Definition 6. Let G be a group and H a normal subgroup of G. We say that G nearly splits over H if there exists a subgroup K of G such that $|G : K|$ is infinite, $|G : HK|$ is finite and $K(G, H \cap K) = 1$.

Lemma. Let G be any group and H a normal subgroup of prime order. Then $\Phi(G) \cap H = 1$ if and only if G splits over H.

Proof. If $\Phi(G) \cap H = 1$, then there exists a maximal subgroup M of G, not containing H. Thus, $G = MH$. Also, since H is of prime order and it is not contained in M, we have $H \cap M = 1$. Therefore, G splits over H.

Conversely, if G splits over H, then there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. Now, if h is any non–trivial element of H, then $< h, K > = HK = G$, but $G \neq K$.

Hence, h is not a nongenerator of G. Thus, $\Phi(G) \cap H = 1$. This completes the proof.

Theorem. Let G be any group and H a normal subgroup of prime order. Then $\lambda(G) \cap H = 1$ if and only if G nearly splits over H.

Proof. Suppose $\lambda(G) \cap H = 1$, so that all non–trivial elements of H are near generators. Then, for any element $1 \neq x \in H$, there exists a proper subset S of G such that $|G : < S >|$ is infinite, but $|G : < S, x >|$ is finite. Now, since H is a normal subgroup of prime order, $< x > = H$ and, therefore,

$$(1) \qquad |G : < S, x >| = |G : < S > < x >| = |G : < S > H|$$

is finite. To prove that G nearly splits over H, we need only show that $K(G, H \cap < S >) = 1$. If not, then $K(G, H \cap < S >) = H$, so that $H \leq < S >$. This implies that

$$|G : < S, x >| = |G : < S > H| = |G : < S >|$$

is infinite, contradicting finiteness of $|G : < S > H|$ in (1).

Conversely, suppose G nearly splits over H; that is, there exists a subgroup K of G such that $|G : HK|$ is finite, $|G : K|$ is infinite and $K(G, H \cap K) = 1$. To prove $\lambda(G) \cap H = 1$, we need to show that every non–trivial element x of H is a near generator; that is, we need to show that there exists a proper subset S of G such that $|G : < S >|$ is infinite, but $|G : < S, x >|$ is finite. We are done if we choose the subset S so that $< S > = K$. This completes the proof.

Note that for the proof of $\lambda(G) \cap H = 1$, we did not make use of the fact that $K(G, H \cap K) = 1$. Thus, nearly splitting is a stronger condition than we need.

Remark. The subgroup H in both the lemma and the theorem must be a proper subgroup of G. Otherwise, $\Phi(G) \cap H = G \neq 1$, and $\lambda(G) \cap H = G \neq 1$.

The Question. Before stating the question, we need two more definitions as well as a proposition.

Definition 7. A subgroup M of a group G is nearly maximal in G if $|G : M|$ is infinite, but $|G : N|$ is finite, for every subgroup N of G properly containing M. That is, M is maximal with respect to being of infinite index in G.

Definition 8. The intersection of all nearly maximal subgroups of a group G is called the upper near Frattini subgroup of G, denoted by $\mu(G)$.

Proposition 2. For every group G, $\lambda(G) \leq \mu(G)$.

Proof. Let $g \in \lambda(G)$, and let M be any nearly maximal subgroup of G. If $g \notin M$, then $< M, g >$ properly contains M, and thus $|G : < M, g >|$ is finite. Now, since $g \in \lambda(G)$, it follows that $|G : M|$ is finite, which is impossible. Hence, g lies in every nearly maximal subgroup of G. Thus, $g \in \mu(G)$, and consequently, $\lambda(G) \leq \mu(G)$.

Note that $\lambda(G)$ and $\mu(G)$ need not coincide. An example of a group G in which $\lambda(G)$ is a proper subgroup of $\mu(G)$ is given by B. Huppert [3, Example 1, p. 162]. However, $\lambda(G)$ and $\mu(G)$ coincide for large classes of groups (see [3]). If $\lambda(G) = \mu(G)$, then their common value is called the near Frattini subgroup of G, denoted by $\psi(G)$.

Finally, we are ready to state the question: Is the statement of the above theorem still true, if $\lambda(G)$ is replaced by $\mu(G)$?

References

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