ONE–TO–ONE CONTINUOUS EXTENSIONS OF ANALYTIC FUNCTIONS

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Let U be the open unit disc in the complex plane. Let $H(U)$ stand for the space of functions analytic on U. Let

$$A = \{g\varepsilon H(U) : g'(0) \neq 0\} \ .$$

For $g\varepsilon A$,

$$g(z) = \sum_{n=0}^{\infty} a_n z^n \ ,$$

following the lead of Walter Rudin (see [1] problem E3325 p.445), we say g has the property P_t if

$$\sum_{n=2}^{\infty} |a_n| n \leq t \ .$$

In this short note we prove the following result which is an extension of the problem E3325 of [1].

Theorem: Let $g\varepsilon A$ have the property P_t for some $t > 0$. Then g is one–to–one and admits a one–to–one continuous extension to the closed unit disc if $t \leq |g'(0)|$. First we prove a lemma.

Lemma: Assume $g \varepsilon A$ and has the property P_t for some $t > 0$. Then g has a continuous extension to the closure of U.

Proof: Let

$$f(z) = (g(z) - g(0))(g'(0))^{-1} .$$

Obviously $f \varepsilon A, f(0) = 0$ and $f'(0) = 1$ and has the property P_s where $s = t|g'(0)|^{-1}$. Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n .$$

Since f has the property P_s, the series

$$z + \sum_{n=2}^{\infty} a_n z^n$$

is absolutely convergent for every z in the unit circle. Now we show that f is continuous on the closed unit disc. Let z, w be in the closed unit disc. We have,

$$
\begin{aligned}
|f(z)-f(w)| &= \left|(z-w)+\sum_{n=2}^{\infty} a_n(z^n-w^n)\right| \\
&= |z-w|\Bigg|1+\sum_{n=2}^{\infty} a_n(z^{n-1}+z^{n-2}w+\cdots \\
&\qquad + zw^{n-2}+w^{n-1})\Bigg| \\
&\le |z-w|\left[1+\sum_{n=2} |a_n|(|z|^{n-1}+|z|^{n-2}|w|+\cdots+|w|^{n-1})\right] \\
&\le |z-w|\left[1+\sum_{n=2}^{\infty} n|a_n|\right] \\
&\le |z-w|(1+s)\ .
\end{aligned}
$$

The next–to–last inequality is true since $|z|\le 1,$ and $|w|\le 1$. From the above it follows that f has a continuous extension to the closed unit disc. Hence, g has a continuous extension to the closed unit disc. Q.E.D.

<u>Proof of the Theorem</u> : Let

$$f=(g(z)-g(0))^{-1}(g'(0))^{-1}\ .$$

Then $f \varepsilon A, f(0) = 0, f'(0) = 1$ and has the property P_t where $t \leq 1$. Since f has continuous extension to the closed unit disc (by the lemma), it is enough to show that f is one–to–one and the extension of f is one–to–one on the closed unit disc. Let

$$f(z) = z + \sum_{n=2}^{\infty} a^n z^n.$$

For z, w in the open unit disc, since

$$|f(z) - f(w)| = \left|(z - w) + \sum_{n=2}^{\infty} a_n (z^n - w^n)\right|$$

$$= |z - w| \left|1 + \sum_{n=2}^{\infty} a_n (z^{n-1} + z^{-2} w + \cdots + w^{n-1})\right|,$$

and since

$$\left|\sum_{n=2}^{\infty} a_n (z^{n-1} + z^{n-2} w + \cdots + w^{n-1})\right| < \sum_{n=2}^{\infty} n|a_n| \leq t \leq 1$$

it follows that f is one–to–one on the open unit disc. Also from the above it follows that $f(z) \neq f(w)$ if one of z, w is in the open unit disc and the other on the unit circle. Now we show that f is one–to–one on the unit circle. By the lemma, f has continuous extension to the closed unit disc. Let $f(U) = \Omega$. If possible assume that z, w are distinct points on the unit circle such that $f(z) = f(w)$. Let $z_n = (1 - n^{-1})z$, $w_n = (1 - n^{-1})w$. Since f is continuous on the closed unit disc, $f(z_n) \to f(z)$ and $f(w_n) \to f(w)$. Since f is a homeomorphism of U onto Ω, $f(z)(= f(w))$ is a boundary point of Ω. Now for each $n \geq 1$, let

$$s_n = \begin{cases} f(z_n) & \text{if } n \text{ is even} \\ f(w_n) & \text{if } n \text{ is odd.} \end{cases}$$

Clearly s_n is a sequence in Ω converging to the boundary point of Ω. Since f^{-1} from Ω onto U is a homeomorphism, the sequence $f^{-1}(s_n)$ must converge to a point of the unit circle. This is a contradiction because by definition of s_n, the sequence $f^{-1}(s_n)$ has two subsequences converging to two different limits. Hence f is one–to–one on the unit circle. It follows that g is one– to–one in the unit disc and has a one–to–one continuous extension to the closed unit disc. Q.E.D.

Remark: For any $t > 1$ there exists a function $g \varepsilon A$ having the property P_t and fails to be one–to–one (refer to (c) of problem

E3325 of [1]). Let $t > 1$. Write $t = 1 + c$. Define

$$g(z) = z - \frac{1}{2}z^2 + \frac{\alpha}{3}z^3 + \frac{\beta}{4}z^4$$

where

$$\alpha = \frac{-72 - 3c}{5} \text{ and } \beta = \frac{8c + 72}{5} \ .$$

Since

$$\sum_{n=2}^{4} n|a_n| = 1 + \alpha + \beta = 1 + c = t \ ,$$

g has the property P_t. Clearly $g \varepsilon A$. Now by the choice of α, β it is easy to verify that $g(1/2) = 0$. Also $g(0) = 0$. Hence g is not one–to–one on the unit disc.

References

1. *The American Mathematical Monthly*, Vol. 96, No. 5, May 1989.