ONE-TO-ONE CONTINUOUS EXTENSIONS
OF ANALYTIC FUNCTIONS

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Let $U$ be the open unit disc in the complex plane. Let $H(U)$ stand for the space of functions analytic on $U$. Let

$$A = \{ g \in H(U) : g'(0) \neq 0 \} .$$

For $g \in A$,

$$g(z) = \sum_{n=0}^{\infty} a_n z^n ,$$

following the lead of Walter Rudin (see [1] problem E3325 p.445), we say $g$ has the property $P_t$ if

$$\sum_{n=2}^{\infty} |a_n| n \leq t .$$

In this short note we prove the following result which is an extension of the problem E3325 of [1].

**Theorem:** Let $g \in A$ have the property $P_t$ for some $t > 0$. Then $g$ is one-to-one and admits a one-to-one continuous extension to the closed unit disc if $t \leq |g'(0)|$. First we prove a lemma.
Lemma: Assume \( g \in A \) and has the property \( P_t \) for some \( t > 0 \).

Then \( g \) has a continuous extension to the closure of \( U \).

Proof: Let

\[
f(z) = (g(z) - g(0))(g'(0))^{-1}.
\]

Obviously \( f \in A \), \( f(0) = 0 \) and \( f'(0) = 1 \) and has the property \( P_s \) where \( s = t|g'(0)|^{-1} \). Let

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\]

Since \( f \) has the property \( P_s \), the series

\[
z + \sum_{n=2}^{\infty} a_n z^n
\]

is absolutely convergent for every \( z \) in the unit circle. Now we show that \( f \) is continuous on the closed unit disc. Let \( z, w \) be in
the closed unit disc. We have,

\[ |f(z) - f(w)| = |(z - w) + \sum_{n=2}^{\infty} a_n(z^n - w^n)| \]

\[ = |z - w|\left[ 1 + \sum_{n=2}^{\infty} a_n(z^{n-1} + z^{n-2}w + \cdots + zw^{n-2} + w^{n-1}) \right] \]

\[ \leq |z - w|\left[ 1 + \sum_{n=2}^{\infty} n|a_n| \right] \]

\[ \leq |z - w|(1 + s) . \]

The next-to-last inequality is true since \(|z| \leq 1, \) and \(|w| \leq 1\). From the above it follows that \(f\) has a continuous extension to the closed unit disc. Hence, \(g\) has a continuous extension to the closed unit disc. Q.E.D.

Proof of the Theorem: Let

\[ f = (g(z) - g(0))^{-1}(g'(0))^{-1} . \]
Then \( f \in A, f(0) = 0, f'(0) = 1 \) and has the property \( P_t \) where \( t \leq 1 \). Since \( f \) has continuous extension to the closed unit disc (by the lemma), it is enough to show that \( f \) is one-to-one and the extension of \( f \) is one-to-one on the closed unit disc. Let

\[
f(z) = z + \sum_{n=2}^\infty a_n z^n.
\]

For \( z, w \) in the open unit disc, since

\[
|f(z) - f(w)| = |z - w + \sum_{n=2}^\infty a_n (z^n - w^n)|
\]

\[
= |z - w| \left(1 + \sum_{n=2}^\infty a_n (z^{n-1} + z^{-2} w + \cdots + w^{n-1})\right),
\]

and since

\[
\left|\sum_{n=2}^\infty a_n (z^{n-1} + z^{-2} w + \cdots + w^{n-1})\right| < \sum_{n=2}^\infty n |a_n| \leq t \leq 1
\]

it follows that \( f \) is one-to-one on the open unit disc. Also from the above it follows that \( f(z) \neq f(w) \) if one of \( z, w \) is in the open unit disc and the other on the unit circle. Now we show that \( f \) is one-to-one on the unit circle. By the lemma, \( f \) has continuous extension
to the closed unit disc. Let \( f(U) = \Omega \). If possible assume that \( z, w \) are distinct points on the unit circle such that \( f(z) = f(w) \). Let \( z_n = (1 - n^{-1})z, \ w_n = (1 - n^{-1})w \). Since \( f \) is continuous on the closed unit disc, \( f(z_n) \to f(z) \) and \( f(w_n) \to f(w) \). Since \( f \) is a homeomorphism of \( U \) onto \( \Omega \), \( f(z)(= f(w)) \) is a boundary point of \( \Omega \). Now for each \( n \geq 1 \), let

\[
s_n = \begin{cases} 
  f(z_n) & \text{if } n \text{ is even} \\
  f(w_n) & \text{if } n \text{ is odd}.
\end{cases}
\]

Clearly \( s_n \) is a sequence in \( \Omega \) converging to the boundary point of \( \Omega \). Since \( f^{-1} \) from \( \Omega \) onto \( U \) is a homeomorphism, the sequence \( f^{-1}(s_n) \) must converge to a point of the unit circle. This is a contradiction because by definition of \( s_n \), the sequence \( f^{-1}(s_n) \) has two subsequences converging to two different limits. Hence \( f \) is one-to-one on the unit circle. It follows that \( g \) is one-to-one in the unit disc and has a one-to-one continuous extension to the closed unit disc. Q.E.D.

Remark: For any \( t > 1 \) there exists a function \( g \in A \) having the property \( P_t \) and fails to be one-to-one (refer to (c) of problem
Let $t > 1$. Write $t = 1 + c$. Define

$$g(z) = z - \frac{1}{2}z^2 + \frac{\alpha}{3}z^3 + \frac{\beta}{4}z^4$$

where

$$\alpha = \frac{-72 - 3c}{5} \quad \text{and} \quad \beta = \frac{8c + 72}{5}.$$  

Since

$$\sum_{n=2}^{4} |a_n| = 1 + \alpha + \beta = 1 + c = t,$$

$g$ has the property $P_t$. Clearly $g \in A$. Now by the choice of $\alpha, \beta$ it is easy to verify that $g(1/2) = 0$. Also $g(0) = 0$. Hence $g$ is not one-to-one on the unit disc.

**References**