SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the editor.

6. *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Prove

$$\sum_{n\le x}\frac{1}{3n-2}=\frac{1}{3}\log(3x-2)+\frac{1}{6}\log 3+\frac{\pi}{6\sqrt{3}}+\frac{\gamma}{3}+O\left(\frac{1}{x}\right)\ ,$$

where log is the natural log and γ is Euler's constant.

Solution by the proposers.

We start with the following lemma.

<u>Lemma</u>.

$$\int_{\frac{1}{3}}^{\infty}\frac{u-[u+\frac{2}{3}]}{u^2}\,du=1-\gamma-\frac{\pi}{2\sqrt{3}}-\frac{1}{2}\log 3\ .$$

<u>Proof</u>.

$$\begin{aligned}
&\int_{\frac{1}{3}}^{\infty}\frac{u-[u+\frac{2}{3}]}{u^2}\,du=\int_{\frac{1}{3}}^{1}\frac{u-[u+\frac{2}{3}]}{u^2}\,du+\sum_{n=1}^{\infty}\int_{n}^{n+1}\frac{u-[u+\frac{2}{3}]}{u^2}\,du\\
&=\int_{\frac{1}{3}}^{1}\frac{u-1}{u^2}\,du+\sum_{n=1}^{\infty}\left(\int_{n}^{n+\frac{1}{3}}\frac{u-[u+\frac{2}{3}]}{u^2}\,du+\int_{n+\frac{1}{3}}^{n+1}\frac{u-[u+\frac{2}{3}]}{u^2}\,du\right)\\
&=\log u+\frac{1}{u}\Big|_{\frac{1}{3}}^{1}+\sum_{n=1}^{\infty}\left(\int_{n}^{n+\frac{1}{3}}\frac{u-[u]}{u^2}\,du+\int_{n+\frac{1}{3}}^{n+1}\frac{u-[u]-1}{u^2}\,du\right)\\
&=\log 3-2+\sum_{n=1}^{\infty}\left(\int_{n}^{n+1}\frac{u-[u]}{u^2}\,du-\int_{n+\frac{1}{3}}^{n+1}\frac{du}{u^2}\right)\\
&=\log 3-2+\int_{1}^{\infty}\frac{u-[u]}{u^2}\,du+\sum_{n=1}^{\infty}\frac{1}{u}\Big|_{n+\frac{1}{3}}^{n+1}
\end{aligned}$$

$$
\begin{aligned}
&= \log 3 - 2 + \int_1^\infty \frac{u - [u]}{u^2}\, du + \sum_{n=1}^\infty \left(\frac{1}{n+1} - \frac{1}{n + \frac{1}{3}} \right) \\
&= \log 3 - 2 + (1 - \gamma) + \left(2 - \frac{\pi}{2\sqrt{3}} - \frac{3}{2} \log 3 \right) \\
&= 1 - \gamma - \frac{\pi}{2\sqrt{3}} - \frac{1}{2} \log 3 \ .
\end{aligned}
$$

The next to last equality follows from

$$
\int_1^\infty \frac{u - [u]}{u^2}\, du = 1 - \gamma \qquad [1]
$$

and

$$
\sum_{n=1}^\infty \left(\frac{1}{n+1} - \frac{1}{n + \frac{1}{3}} \right) = 2 - \frac{\pi}{2\sqrt{3}} - \frac{3}{2} \log 3 \qquad [2] \ .
$$

To prove the result, we have by Euler's summation formula [1] that

$$
\begin{aligned}
&\sum_{n \le x} \frac{1}{3n-2} = 1 + \int_1^x \frac{dt}{3t-2} - 3 \int_1^x \frac{t - [t]}{(3t-2)^2}\, dt + \frac{[x] - x}{3x - 2} \\
&= \frac{1}{3} \log(3x - 2) + 1 - 3 \int_1^x \frac{t - [t]}{(3t-2)^2}\, dt + O\left(\frac{1}{x} \right) \\
&= \frac{1}{3} \log(3x - 2) + 1 - 3 \int_1^\infty \frac{t - [t]}{(3t-2)^2}\, dt + 3 \int_x^\infty \frac{t - [t]}{(3t-2)^2}\, dt + O\left(\frac{1}{x} \right) \\
&= \frac{1}{3} \log(3x - 2) + 1 - 3 \int_1^\infty \frac{t - [t]}{(3t-2)^2}\, dt + O\left(\frac{1}{x} \right) \ .
\end{aligned}
$$

Next,

$$
\begin{aligned}
\int_1^\infty \frac{t - [t]}{(3t-2)^2}\, dt &= \frac{1}{9} \int_1^\infty \frac{t - [t]}{(t - \frac{2}{3})^2}\, dt \\
&= \frac{1}{9} \int_{\frac{1}{3}}^\infty \frac{u + \frac{2}{3} - [u + \frac{2}{3}]}{u^2}\, du \\
&= \frac{1}{9} \int_{\frac{1}{3}}^\infty \frac{u - [u + \frac{2}{3}]}{u^2}\, du + \frac{2}{27} \int_{\frac{1}{3}}^\infty \frac{du}{u^2} \\
&= \frac{1}{9} \int_{\frac{1}{3}}^\infty \frac{u - [u + \frac{2}{3}]}{u^2}\, du + \frac{2}{9} \ .
\end{aligned}
$$

Thus by the lemma,

$$\int_1^\infty \frac{t-[t]}{(3t-2)^2}\,dt = \frac{1}{9}\big(1-\gamma-\frac{\pi}{2\sqrt{3}}-\frac{1}{2}\log 3\big)+\frac{2}{9}$$
$$=\frac{1}{3}-\frac{\gamma}{9}-\frac{\pi}{18\sqrt{3}}-\frac{1}{18}\log 3\ ,$$

so

$$\sum_{n\le x}\frac{1}{3n-2}=\frac{1}{3}\log(3x-2)$$
$$+1-3\big(\frac{1}{3}-\frac{\gamma}{9}-\frac{\pi}{18\sqrt{3}}-\frac{1}{18}\log 3\big)+O\big(\frac{1}{x}\big)$$
$$=\frac{1}{3}\log(3x-2)+\frac{1}{6}\log 3+\frac{\pi}{6\sqrt{3}}+\frac{\gamma}{3}+O\big(\frac{1}{x}\big)\ .$$

References

1. T. M. Apostol, *Introduction to Analytic Number Theory*, Springer–Verlag, 1984, pp. 56,54.
2. D. E. Knuth, *The Art of Computer Programming: Fundamental Algorithms*, Vol. 1, Second Edition, Addison–Wesley, p. 94.

9. *Proposed by Robert E. Shafer, Berkeley, California.*

Let p be a prime. Prove that

$$(p-1)! \equiv (p+1)\big(1^{p-1}+2^{p-1}+3^{p-1}+\cdots+(p-1)^{p-1}\big) \pmod{p^2}\ .$$

Solution by Bob Prielipp, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin.

The problem as stated above is incorrect. Letting $p=2$ yields a counterexample. In the remainder of our solution we shall assume that p is an odd prime number.

We shall use the following known result. If p is an odd prime number, then

$$(*) \quad 1^{p-1} + 2^{p-1} + \cdots + (p-1)^{p-1} \equiv p + (p-1)! \pmod{p^2} .$$

[For a proof of (*), see exercise 5 and its solution on p. 211 of Sierpinski, Elementary Theory of Numbers, Hafner Publishing Company, New York, 1964.] We may now construct the following collection of equivalent congruences.

$$(p-1)! \equiv (p+1)(1^{p-1} + 2^{p-1} + \cdots + (p-1)^{p-1}) \pmod{p^2}$$

$$(p-1)! \equiv (p+1)(p + (p-1)!) \pmod{p^2} \qquad [\text{by } (*)]$$

$$(p-1)! \equiv p^2 + p(p-1)! + p + (p-1)! \pmod{p^2}$$

$$p(p-1)! \equiv -p \pmod{p^2}$$

$$(p-1)! \equiv -1 \pmod{p} .$$

But $(p-1)! \equiv -1 \pmod{p}$ follows from Wilson's Theorem so our solution is complete.

Also solved by the proposer.

12. *Proposed by Robert E. Shafer, Berkeley, California.*

Evaluate

$$\int_0^1 e^{1/\log x}\, dx ,$$

where log denotes the natural log.

Solution by Mark Ashbaugh, University of Missouri, Columbia, Missouri.

The integral evaluates to $2K_1(2) \approx .2797318$ where K_1 denotes one of the two standard modified Bessel functions of order 1. It would be surprising though perhaps not impossible for there to be a more elementary expression for the value of this integral.

Let I denote the given integral. Then under the substitution $t = -1/\log x$ it becomes

$$\begin{aligned} I &= \int_0^\infty t^{-2} e^{-(t+1/t)}\, dt \\ &= \int_0^1 t^{-}2e^{-(t+1/t)}\, dt + \int_1^\infty t^{-2} e^{-(t_1/t)}\, dt\ . \end{aligned}$$

Now change variables to $s = 1/t$ in the first of these integrals to arrive at

$$\begin{aligned} I &= \int_1^\infty e^{-(s+1/s)}\, ds + \int_1^\infty t^{-2} e^{-(t+1/t)}\, dt \\ &= \int_1^\infty (1 + t^{-2}) e^{-(t+1/t)}\, dt\ . \end{aligned}$$

Finally, put $t = e^u$ in this integral, obtaining

$$I = 2\int_0^\infty \cosh u e^{-2\cosh u}\, du\ .$$

Since one of the standard integral representations for the modified Bessel function $K_\nu(z)$ is

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t} \cosh(\nu t)\, dt$$

[Abramowitz and Stegun, p. 376, eq. 9.6.24] we can identify the value of our integral as

$$
\begin{aligned}
I &= 2K_1(2) \\
&= 2\left[\gamma I_1(2) + \frac{1}{2} - \sum_{n=0}^{\infty} \frac{H_n + H_{n+1}}{n!(n+1)!}\right]
\end{aligned}
$$

where $\gamma \approx .5772157$ is Euler's constant,

$$H_n \equiv \sum_{m=1}^{n} \frac{1}{m}$$

with the convention that $H_0 = 0$, and I_1 represents the other standard modified Bessel function of order 1 and thus, in particular,

$$I_1(2) = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} \ .$$

In an analogous fashion one can show that the function $I(a)$ defined by the integral

$$I(a) = \int_0^1 e^{a/\log x}\, dx \quad \text{for } a > 0$$

is given by

$$I(a) = 2\sqrt{a}K_1(2\sqrt{a}) \ .$$

The sequence of substitutions to be used is now $t = -\sqrt{a}/\log x$, $s = 1/t$, and $t = e^u$.

Also solved by the proposer.