LONG–AGO PROBLEMS FOR THE STUDENT OF TODAY

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The famous problems of antiquity are three in number and concern remarkable constructions. These three, the trisecting of an angle, the duplicating of a cube, and the squaring of a circle, proved the source of devoted geometric pursuit and serendipitous findings to generations of mathematicians. Only in the nineteenth century were all resolved, each one in the negative.

Figure 1. Famous Problems of Antiquity

These longstanding problems all imply extended problems of a highly generalized kind. Thus, can the general angle be divided into four or into five "equal" parts? Can a cube be triplicated or a circle triangulated? Let us now take another look at the THREE FAMOUS PROBLEMS OF ANTIQUITY, this time in the generalized setting.

Can Any Angle Be Pentasected, ... ?

Although any angle can be bisected by use of the Euclidean instruments, namely the unmarked straightedge and the compass, not all angle divisions are possible. Quite likely, the geometry student is well aware of the fact that a general method of angle trisection is impossible. However, what can be said in reference to higher orders of division? The follow–up questions of quadrisection, pentasection, and of multisection rarely seem to surface in the typical discussion of constructions. Yet the answers to such far–reaching questions prove within the grasp of those studying elementary geometry.

The ancient Greeks, originators of the famous problems of antiquity, scarcely touched on the generalized subject, even by way of inquiry. Perhaps their lack of success in the case for angle trisection served as a brick wall to further pursuits in the overall area.

Any angle can be divided into n congruent ("equal") parts if n is a power of 2 simply by repeated bisecting. Hence, the results

n	n–section of angle
2	possible
3	impossible
4	possible
5	?

But what can be said about succeeding entries in the table? For example, can the general angle be partitioned into five "equal" parts? Or six? Or seven?

Let us note first that if the angle of 360 degrees cannot be n–sected, then no method of angle n–section exists. But the problem of n–section in reference to a 360 degree angle is precisely the problem of regular polygon constructibility. For example, if the 360 degree angle can be partitioned into n "equal" parts $(n > 2)$, these parts thus give rise to appropriate central angles of regular n–sided polygons. Very simply, the regular n–sided polygon is constructible if and only if a 360 degree angle can be n–sected $(n > 2)$.

The problem accordingly is to establish the values of n for which the regular n–sided polygon can be constructed. Such a resolution stems from the works of Gauss in 1801 and his slightly later successors. Without loss of generality, our quest will be restricted to regular polygons of an odd number of sides. Note, for example, that if an angle cannot be divided into x "equal" parts, neither can it be divided into $2x$ "equal" parts. The standard of constructibility reads as follows:

> A regular polygon of an odd number of sides, say n, is constructible if and only if n is a prime of the form $2^k + 1$ or the product of distinct primes all of this form.

Some refer to this as the Gaussian standard. Moreover, primes of the form $2^k + 1$ are often called Fermat primes; the exponent k is of necessity a power of 2. Only five primes of the form $2^k + 1$ are today known, namely, 3, 5, 17, 257, and 65537.

The regular five–sided polygon can be constructed as 5 is a prime of the form $2^k + 1$, that is, $5 = 2^2 + 1$. Yet the regular 25–sided polygon is not constructible as 25 cannot be expressed as the product of <u>distinct</u> Fermat primes. Note that $25 = (5)(5)$ and that the Fermat prime factors are alike. Should any angle be pentasectible, then the <u>repeated</u> dividing of an angle by five would be possible. In this case, the regular 25–gon, 125–gon, 625–gon, etc. would be constructible with the Euclidean tools. All of these numbers violate the distinctness of the prime factors in the Gaussian standard above.

Note also that the regular seventeen–sided polygon can be constructed as 17 is a prime of the form $2^k + 1$, that is, $2^4 + 1$. Accordingly, a 360 degree angle can be hepta–deca–sected, forming a central angle measuring exactly $21\,\frac{3}{17}$ degrees in the process. However, this angle of $21\,\frac{3}{17}$ degrees cannot be hepta–deca–sected itself as this would give rise to the central angle of a 289–sided regular polygon. The impossibility stems from the fact that $289 = (17)(17)$, a factorization in which the Fermat primes are not distinct. In passing, the construction of the regular heptadecagon is the achievement of the young mathematician Carl Friedrich Gauss (in the year 1796 at the age of nineteen).

Figure 2.
–a regular 5–sided polygon which is constructible and
a regular 25–sided polygon which is not constructible–
Hence, there is no general method of angle pentasection.

By a similar strategy, no general method of angle n–section exists for a prime of the form $2^k + 1$ nor for any product of primes all of this form. Otherwise, successive n–sections would require a repetition of Fermat primes as factors in the dividing of a 360 degree angle. If n is simply an odd number other than a Fermat prime or the product of such primes, n–section is obviously impossible. These are unacceptable numbers by the Gaussian standard. In such a case, the general angle cannot be heptasected (divided by seven) nor uni–deca–sected (divided by eleven) nor n–sected where n is any odd integer greater than 1.

Again, even multiples of any odd number n (greater than 1) give rise to non–constructible divisions also. Obviously, if $(2n)$–section is possible for the general angle, so is n–section. As a consequence of this, the only multi–sections possible are those involving repeated bisections. Hence, in the case for generalized constructible n–sections, n must be a power of 2.

It is important to keep in mind that the impossibility of generalized n–section does not rule out the construction of the regular n–sided polygon. As shown above, the regular pentagon is clearly constructible, yet no general method of angle pentasection exists. Too, the equilateral triangle is constructible in spite of the fact that no general method of angle trisection exists.

Figure 3.

No general method of pentasecting an angle exists.

All of the multisections of this discussion have been in reference to angles and not to line segments. As is well–known, a line segment can be divided into n "equal" parts for any natural number n.

Can a Cube Be Triplicated, . . . ?

Even as the famous trisection problem from antiquity gives rise to generalized questions, so does the problem of duplicating a cube. A cube can be duplicated only if its edge length can be multiplied by the cube root of 2. It is easily shown that $\sqrt[3]{2}$ denotes a non–constructible length, and as a consequence, the duplicating of a cube proves impossible.

By constrast, the square root of any given length can be constructed. The availability thus of $\sqrt{2}$ readily permits the duplicating of a square, that is, the constructing of a square having twice the area of a given square. A vastly different situation exists however in the case for cube roots.

In the event length x is restricted to the positive integers, it is possible to construct $\sqrt[3]{x}$ only as x is an exact third power. Hence, $\sqrt[3]{8}$ is constructible (being given a unit length) but $\sqrt[3]{2}$ is not. The problem of triplicating a cube with edge x necessitates constructing a length y such that $y = \sqrt[3]{3}(x)$. But a length equal to $\sqrt[3]{3}$ is not constructible (by the rule above). In light of this, it is apparent that a cube cannot be triplicated. That is, no construction permits constructing a cube having three times the volume of a given cube.

The generalized question has a simple answer. No valid Euclidean construction exists by which a cube's volume can be multiplied by any natural number other than an exact third power.

Figure 4.

No method of triplicating a cube exists.

Can A Circle Be Triangulated, . . . ?

In 1882, C. F. Lindemann proved that the squaring of a circle is impossible. Such a construction demanded the constructing of π (pi). In demonstrating the fact that π could not occur as a root of an algebraic equation with integral coefficients, its non–constructibility followed. The longstanding problem of the quadrature of a circle was at last resolved. Pi (π) falls into a class of numbers called transcendental; moreover, no transcendental length is constructible.

But, can an equilateral triangle be constructed whose area is the same as that of a given circle? Or can a regular pentagon be constructed with the same equal area property? The answer hinges on the fact that any polygon can be transformed into a triangle of equal area, and moreover, that any triangle can be squared.

Consider any polygon of n sides where n is greater than 3. It is possible to construct a polygon of $(n-1)$ sides so that the two polygons will have the same area. An illustration of this technique is provided by the pentagon on the following page.

Figure 5.

Given: pentagon PQRST

Problem: to construct a quadrilateral having the same area as the given pentagon

Steps: Extend side $\overline{QR}$ to intersect a parallel through S to diagonal $\overline{TR}$. Let the point of intersection be U. Then pentagon PQRST will have the same area as quadrilateral PQUT.

Proof: Both the pentagon PQRST and the quadrilateral PQUT have the region PQRT in common.

Area PQRST = area PQRT + area RST.

Area PQUT = area PQRT + area RUT.

Moreover, triangles RST and RUT have equal areas as they have the same base $\overline{RT}$ and equal altitudes through S and U respectively (recall that $\overline{SU}$ and $\overline{TR}$ are parallel).

Therefore, pentagon PQRST has the same area as quadrilateral PQUT.

By a repetition of this procedure, any polygon can eventually be reduced to a triangle of equal area. As any triangle can be squared, it follows that any polygon can be squared. Clearly then, no circle can be transformed into a polygon, regular or not, of equal area. Otherwise, a squaring of the polygon would in effect constitute a squaring of the circle. Not only is it impossible to square a circle, it is also impossible to construct a regular polygon of any kind having the same area as a given circle.

Figure 6.

No method of triangulating a circle exists.

It should be kept in mind that all Euclidean constructions must be performed in a finite number of steps. Should a regular polygon be inscribed in a circle and the number of sides of the polygon repeatedly increased (still preserving the regular and inscribed features), the polygonal area would approach the circle's area as a limit. As any of these inscribed polygons can be squared, it is evidently the case that many find approximations exist in the case for squaring a circle.

An intriguing outgrowth of this problem is one of a three dimensional kind. A step–up dimensionally from the problem of squaring a circle is that of cubing a sphere. As this latter construction requires the constructing of π, it too is impossible.

Figure 7.
No method of cubing a sphere exists. The reader may wish to pursue this problem in reference to the other regular polyhedra.

The breakthroughs which resolved the THREE FAMOUS PROBLEMS OF ANTIQUITY were advances of a far–reaching kind. Theorems, powerful but concise, were of sufficient generality to dispose not only of precise, highly specific problems but more. Much more! Insightful relationships encompassed such results as

> "If a cubic equation with integral coefficients has no rational roots, then none of its roots are constructible," and

> "No transcendental length is constructible."

Remarkable too is the unity of mathematics demonstrated by an algebraic disposition of that which is purely geometric. Brought to mind in the glowing picture are such names as those of Galois, Wantzel, Hermite, Lindemann, and Gauss. Open doors, as a consequence of the advances above, thus led the solver ever farther along the paths of abstraction, rigorization, and generalization.

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