ON THE COMMON ZEROES OF FINITE BLASCHKE PRODUCTS

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In [1] it has been shown that if K is any non–empty closed subset of the complex plane without any interior, and $P_1, P_2, \cdots, P_n$ are polynomials with complex coefficients such that any complex linear combination of P_i's has a zero in K, then P_i's must have a common zero in K. In this short note we prove a similar result for finite Blaschke products on the unit disc U of the complex plane. More precisely, if $\mathcal{B}$ is a finite dimensional vector space of analytic functions on U consisting of a basis of finite Blaschke products, and K is a non–empty closed set in $U - \{0\}$ with empty interior such that each element of $\mathcal{B}$ has a zero in K, then there exists a z_0 in K such that $g(z_0) = 0$ for every g in $\mathcal{B}$. Refer to [2], for results on Blaschke products.

Theorem. Let $\{k_i\}_{i=1,2,\cdots,n}$ be a sequence of positive integers and, $\{l_i\}_{i=1,2,\cdots,n}$ be sequences of non–negative integers. Let K be a non–empty closed set in $U - \{0\}$ such that $K^0 = \Phi$. For each $i, 1 \leq i \leq n$, let $S_i = \{\alpha_{ij} \in U - \{0\} : 1 \leq j \leq k_i\}$ and

$$B_i(z) = z^{l_i} \prod_{j=1}^{k_i} \frac{\alpha_{ij} - z}{1 - \overline{\alpha_{ij}} z} \frac{|\alpha_{ij}|}{\alpha_{ij}} \text{ for all } z \text{ in } U \ .$$

Further assume that

$$\sum_{i=1}^{n} \omega_i B_i$$

has a zero in K for any given complex numbers $\omega_i, 1 \leq i \leq n$. Then

$$\bigcap_{i=1}^{n} S_i \cap K \neq \Phi.$$

As mentioned in the introduction we prove the above theorem using the following lemma.

Lemma. Let K be any non–empty closed subset of the complex plane such that $K^0 = \Phi$. Let $P_i, 1 \leq i \leq k$ be non–constant polynomials with complex coefficients. Further assume

$$\sum_{i=1}^{n} \omega_i P_i$$

has a zero in K for any complex numbers $\omega_i, 1 \leq i \leq k$. Then there exists a z_0 in K such that $P_i(z_0) = 0$ for $1 \leq i \leq n$.

Proof. Refer for Lemma 1.1 of [1].

Proof of the theorem.

Let

$$P_i(z) = z^{l_i} \prod_{j=1}^{k_i} (\alpha_{ij} - z) \frac{|\alpha_{ij}|}{\alpha_{ij}} \prod_{\substack{t=1 \\ t \neq i}}^{n} \left(\prod_{j=1}^{k_t} (1 - \overline{\alpha_{tj}} z) \right)$$

for each $i, 1 \leq i \leq n$. Let $c_1, c_2, \cdots, c_n$ be any arbitrary complex numbers. Since $c_1 B_1 + c_2 B_2 + \cdots + c_n B_n$ has at least one zero in K, it follows that $c_1 P_1 + c_2 P_2 + \cdots + c_n P_n$ which is the numerator of $c_1 B_1 + c_2 B_2 + \cdots + c_n B_n$, has at least one zero in K. Thus $P_i, 1 \leq i \leq n$ satisfies the hypothesis of the lemma. Hence by the lemma $P_i, 1 \leq i \leq n$ have at least one common zero in K. Since $|\alpha_{ij}|^{-1} > 1$ for each i and j, and K is contained in $U - \{0\}$, the polynomials

$$\prod_{j=1}^{k_i} (\alpha_{ij} - z)$$

for each $i, 1 \leq i \leq n$, have a common zero in K. This obviously implies that

$$\bigcap_{i=1}^{n} S_i \cap K \neq \Phi. \quad Q.E.D.$$

References

1. R. Garimella and N. V. Rao, "Closed Subspaces of Finite Codimension in Some Function Algebras," *Proc. Amer. Math. Soc.*, 101(1987), 657–661.

2. J. B. Garnett, "Bounded Analytic Functions," Academic Press, New York, 1981.