EXTENSIONS OF TOPOLOGIES

T. L. Hicks

University of Missouri–Rolla

Abstract. The m–extension of a topology is defined, some of its properties are studied, and its relation to a simple extension is noted. Also, an associated closure operation on the lattice of all topologies on a fixed set is studied.

Let X be a set, $m = \{M_\nu : \nu \in I\}$ a non–empty collection of subsets of X and, for each $\nu \in I$, t_ν a topology for M_ν. We define a collection $t(m)$ of subsets of X as follows: $V \in t(m)$ if and only if for each $\nu \in I$, $V \cap M_\nu \in t_\nu$. It is clear that $t(m)$ is a topology for X.

Unless otherwise specified we adopt the terminology of Gaal [2].

Theorem 1. $t(m)$ is the strongest topology for X such that $t(m) \leq t_\nu$ on every M_ν.

Proof. If $A \subseteq M_\nu$ and A is a member of the subspace topology $t(m) \cap M_\nu$ on M_ν, then $A = U \cap M_\nu$ where $U \in t(m)$. Thus $A \in t_\nu$. Suppose t' is a topology for X such that $t' \leq t_\nu$ on every M_ν. If

$V \in t'$ and $\nu \in I$, then $V \cap M_\nu \in t' \cap M_\nu \leq t_\nu$. Hence $V \in t(m)$.

Theorem 2. If there exists a topology t' for X such that $t' = t_\nu$ on every M_ν, then $t(m) = t_\nu$ on every M_ν and $t(m)$ is the strongest topology for X with this property.

Proof. If such a topology t' exists, then $t' \leq t(m)$ by Theorem 1. For $\nu \in I$, $t_\nu = t' \cap M_\nu \leq t(m) \cap M_\nu \leq t_\nu$. Hence $t(m) = t_\nu$ on M_ν.

In the remainder of this paper we suppose that t is a topology for X and, for each $\nu \in I$, t_ν is the subspace topology $t \cap M_\nu$. In this case, $t(m)$ is called the m–extension of t. It follows from Theorem 2, that the m–extension of t is the strongest topology on X that agrees with t on every M_ν. Thus $t \leq t(m)$.

Theorem 3. The following properties hold.

1) $t(m)$ is discrete on $X - \bigcup\{M_\nu : \nu \in I\}$, and if $X \in m$, $t = t(m)$.

2) A subset B of X is $t(m)$ closed if and only if, for every $\nu \in I$, $B \cap M_\nu$ is closed in (M_ν, t_ν).

3) If (X, t) is a compact Hausdorff space, then $t = t(m)$ if and only if $t(m)$ is compact.

4) If m is the collection of all closed compact subsets of (X, t), then $t = t(m)$ if and only if X is a k–space.

Proof. 1 and 2 follow easily. For 3, suppose $t(m)$ is compact. $t \le t(m)$. $t = t(m)$ since it is not possible to have a Hausdorff topology strictly weaker than a compact topology. For 4, recall that a k–space is one that satisfies the condition: if a subset A of X intersects each closed compact set in a closed set, then A is closed.

In [1], Levine defined a topology $t^* > t$ to be a simple extension of t if there exists a subset A of X such that $A \notin t$ and $t^* = \{U \cup (V \cap A) : U, V \in t\}$. We write $t^* = t(A)$. Many of his results dealt with the case $X - A \in t$. We need the following results which appear in [1].

(a) t and $t(A)$ agree on A and also on $X - A$.

(b) A is closed in (X, t) if and only if A is closed in $(X, t(A))$.

Theorem 4.

(1) $t(A)$ is the weakest topology on X such that $t < t(A)$ and $A \in t(A)$.

(2) Let $m = \{A, X - A\}$. $t(A) = t(m)$ if and only if $X - A \in t$.

Proof. (1) If $t' > t$ and $A \in t'$, let $N \in t(A)$. $N = U \cup (V \cap A)$

where $U, V \in t$. Since $A \in t'$ and $t < t'$, $N \in t'$.

(2) By (a), t and $t(A)$ agree on every set of m. Hence $t(A) \leq t(m)$ by Theorem 1. Suppose $X - A \in t$. If $U \in t(m)$, $U \cap A \in t \cap A = t(A) \cap A$. Since $A \in t(A)$, $U \cap A \in t(A)$. Also, $X - A \in t(A)$ implies that $U \cap (X - A) \in t(A)$. Therefore, $U = (U \cap A) \cup (U \cap X - A) \in t(A)$.

$X - A \in t(m)$, so that if $t(m) = t(A)$, $X - A \in t(A)$.

Theorem 5.

(1) $t \leq t(m)$ and, in general, $t \neq t(m)$. In fact, if m consists only of singleton subsets, then $t(m)$ is the discrete topology.

(2) $(t(m))(m) = t(m)$ and if $m \subseteq n$, then $t(m) = (t(m))(n)$.

(3) If $m \subseteq n$, $t(m) \geq t(n)$.

(4) If $X \in m$, $t(m) = t$.

(5) If $m = \{M_\nu : \nu \in I\}$ is a collection of t open sets such that $\bigcup\{M_\nu : \nu \in I\} = X$, then $t = t(m)$.

(6) If u is a topology for X and $t \leq u$, then $t(m) \leq u(m)$.

Proof. Since the properties are easily established, we only offer a proof for 2. $t(m) \leq (t(m))(n)$ by 1. If $A \in (t(m))(n)$ and $m \subseteq n$, then $A \cap M_\nu \in t(m) \cap M_\nu = t_\nu$ for every $\nu \in I$. Thus $A \in t(m)$

and equality holds.

Definition. By a closure operation on a partially ordered set $(P, \leq)$ we mean a mapping $\psi : P \to P$ such that:

(1) $x \leq y$ implies $\psi(x) \leq \psi(y)$.

(2) $x \leq \psi(x)$ for every $x \in P$.

(3) $\psi(\psi(x)) = \psi(x)$ for every $x \in P$.

Suppose $m = \{M_\nu : \nu \in I\}$ is fixed. If we look at properties 1, 2, and 6 in the above theorem we see that m defines a closure operation $(t \to t(m))$ on the lattice P^* of all topologies on X.

Theorem 6. Suppose $\{t_\alpha : \alpha \in J\}$ is a non–empty collection of topologies on X.

(1) $(\operatorname*{glb}_\alpha t_\alpha)(m) \leq \operatorname*{glb}_\alpha t_\alpha(m) = (\operatorname*{glb}_\alpha t_\alpha(m))(m)$.

(2) $(\operatorname*{lub}_\alpha t_\alpha)(m) = (\operatorname*{lub}_\alpha t_\alpha(m))(m)$.

Proof. (1) The equality in (1) is a special case of a Theorem of Ward [4, p. 68 or 5], but the reader can easily verify (1).

(2) For each α, $t_\alpha \leq \operatorname*{lub}_\alpha t_\alpha$ so that $t_\alpha(m) \leq (\operatorname*{lub}_\alpha t_\alpha)(m)$. Hence $\operatorname*{lub}_\alpha t_\alpha(m) \leq (\operatorname*{lub}_\alpha t_\alpha)(m)$ and consequently $(\operatorname*{lub}_\alpha t_\alpha(m))(m) \leq (\operatorname*{lub}_\alpha t_\alpha)(m)$. Also, $\operatorname*{lub}_\alpha t_\alpha \leq \operatorname*{lub}_\alpha t_\alpha(m)$ implies that $(\operatorname*{lub}_\alpha t_\alpha)(m) \leq$

$(\operatorname*{lub}_{\alpha} t_\alpha(m))(m)$.

Example 1. The $\leq$ in (1) of Theorem 6 may be $<$. Put $X = \{a, b, c, d\}$ and $m = \{M\}$ where $M = \{a, b, c\}$. Let $t = \{\{c\}, X, \emptyset\}$ and $u = \{\{d, c\}, X, \emptyset\}$. Then $(t \cap u)(m) < t(m) \cap u(m)$.

Corollary.

(1) $(\operatorname*{lub}_{\alpha} t_\alpha)(m) = \operatorname*{lub}_{\alpha} t_\alpha(m)$ if and only if $\operatorname*{lub}_{\alpha} t_\alpha(m)$ is a fixed point of m.

(2) If $\operatorname*{lub}_{\alpha} t_\alpha$ is a fixed point of m, then so is $\operatorname*{lub}_{\alpha} t_\alpha(m)$.

Proof. In general, $\operatorname*{lub}_{\alpha} t_\alpha(m) \leq (\operatorname*{lub}_{\alpha} t_\alpha)(m) = (\operatorname*{lub}_{\alpha} t_\alpha(m))(m)$. ∎

Theorem 7. If $(\operatorname*{glb}_{\alpha} t_\alpha) \cap M_\nu = \operatorname*{glb}_{\alpha} (t_\alpha \cap M_\nu)$ for every $\nu \in I$, then $\operatorname*{glb}_{\alpha} t_\alpha(m) = (\operatorname*{glb}_{\alpha} t_\alpha)(m)$.

Proof. Suppose $U \in t_\alpha(m)$ for every α and let ν be fixed. Since $U \cap M_\nu \in t_\alpha \cap M_\nu$, $U \cap M_\nu \in \operatorname*{glb}_{\alpha} (t_\alpha \cap M_\nu) = (\operatorname*{glb}_{\alpha} t_\alpha) \cap M_\nu$. Hence $U \in (\operatorname*{glb}_{\alpha} t_\alpha)(m)$ and $\operatorname*{glb}_{\alpha} t_\alpha(m) \leq (\operatorname*{glb}_{\alpha} t_\alpha)(m)$.

Question. Is every closure operation on P^* of the above form? That is, given a closure operation ψ on P^* does there exist a collection $m = \{M_\nu\}$ of subsets of X such that $\psi(t) = t(m)$ for every $t \in P^*$?

Theorem 8. Suppose ψ is given. If there exists a class $m = \{M_\nu : \nu \in I\}$ such that $\psi(t) = t(m)$ for every $t \in P^*$, then $\psi(t) = t(m^*)$ for every $t \in P^*$ where $m^* = \{M \subseteq X : t$ and $\psi(t)$ agree on M for every $t \in P^*\}$.

Proof. Since $t \leq \psi(t) = t(m)$, t and $\psi(t)$ agree on every $M \in m$. Hence $m \subseteq m^*$ and consequently $t(m) \geq t(m^*)$. By Theorem 2, $t(m^*) \geq \psi(t)$. We have $\psi(t) = t(m) \geq t(m^*) \geq \psi(t)$ for every $t \in P^*$.

Example 2. This example shows that the answer to the above question is no. Let X denote an infinite set, t^* the discrete topology, t' the indiscrete topology, and u the topology of finite complements. Define ψ as follows: $\psi(t) = u$ if $t \leq u$ and $\psi(t) = t^*$ otherwise. The reader can easily verify that ψ is a closure operation on P^*. We show that m^*, defined in the above theorem, consists of all singleton subsets. Then since $t(m^*)$ is the strongest topology that agrees with t on every $M \in m^*$, $t(m^*) = t^*$ for every $t \in P^*$ and consequently $\psi(t) \neq t(m^*)$ for any $t \leq u$.

Suppose $x \neq y$ and let $A = \{x, y\}$. It suffices to prove that $\psi(t') = u$ and t' do not agree on A. $A \cap (X - \{y\}) = \{x\}$ so that

$\{x\} \in u \cap A$. $\{x\} \notin t' \cap A$.

Problem. Characterize every closure operation on P^*.

References

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