ON THE LOWER NEAR FRATTINI SUBGROUPS OF AMALGAMATED FREE PRODUCTS OF GROUPS I

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1. **Introduction.** In this paper, we briefly reintroduce the Frattini subgroup $\Phi(G)$, the lower near Frattini subgroup $\lambda(G)$, the upper near Frattini subgroup $\mu(G)$, the near Frattini subgroup $\psi(G)$, of a group G, as well as the amalgamated free products of groups. Also, we prove a generalization of a lemma by C. Y. Tang, and its exact analog for the lower near Frattini subgroups. Finally, we propose two questions for readers.

2. **Notation and Definitions.** Our notation will be standard.

Definition 1. An element g of a group G is a *nongenerator* of G if for every subset S of G such that $< S, g > = G$, then $< S > = G$.

Definition 2. The set of all nongenerators of a group G forms a characteristic subgroup called the *Frattini subgroup* of G, denoted by $\Phi(G)$. The intersection of all maximal proper subgroups of G

coincides with $\Phi(G)$. If there are no maximal proper subgroups of G, then $\Phi(G) = G$.

For more results concerning $\Phi(G)$ see [9].

Definitions involving near Frattini subgroups are due to J. B. Riles [7].

Definition 3. An element g of a group G is a *near generator* of G if there is a subset S of G such that $|G :< S > |$ is infinite, but $|G :< g, S > |$ is finite.

Definition 4. An element g of a group G is a *non–near generator* of G if $S \subseteq G$ and $|G :< g, S > |$ is finite imply that $|G :< S > |$ is finite.

Definition 5. By [1, Proposition 1] the set of all non–near generators of a group G is a subgroup of G. This subgroup is called the *lower near Frattini* subgroup of G, denoted by $\lambda(G)$.

Note 1. $\lambda(G)$ is a characteristic subgroup of G, since every automorphism of G maps a non–near generator of G to a non–near generator of G.

Definition 6. A subgroup M of a group G is *nearly maximal* in G if $|G : M|$ is infinite, but $|G : N|$ is finite, for every subgroup N

of G properly containing M. That is, M is maximal with respect to being of infinite index in G.

Definition 7. The intersection of all nearly maximal subgroups of a group G is called the *upper near Frattini subgroup* of G, denoted by $\mu(G)$.

Note 2. $\mu(G)$ is a characteristic subgroup of G, since every automorphism of G permutes the nearly maximal subgroups of G among themselves.

Definition 8. Let G be any group. If $\lambda(G) = \mu(G)$, then their common value is called the *near Frattini* subgroup of G, denoted by $\psi(G)$.

Definition 9. If H is a subgroup of a group G, then

$$K(G, H) = \bigcap_{g \in G} g^{-1} H g$$

is called the *core* of H in G. $K(G, H)$ is the unique largest normal subgroup of G contained in H.

3. **Amalgamated Free Products of Groups.** Free products of groups with amalgamated subgroups was published first in 1927 by the German mathematician O. Schreier [8]. Hanna

Neumann generalized Schreier's original results, and her work was published in two separate papers in 1948 [4] and 1949 [5]. B. H. Neumann in his 1954 paper [3], redefined, studied, and applied Schreier's and H. Neumann's results to a number of problems in group theory. In this paper we adopt the viewpoint of B. H. Neumann's paper and the following usages for free products of groups with amalgamations.

Let Γ be an indexing set of cardinality greater than one, and let G be a group with a set S of generators. Suppose that

$$S = \bigcup_{\gamma \in \Gamma} S_\gamma$$

is a union of subsets S_γ. Set $G_\gamma = < S_\gamma >$ for each γ, and let R_γ be a set of defining relations for G_γ. If

$$R = \bigcup_{\gamma \in \Gamma} R_\gamma$$

is a set of defining relations for G, then G is the *generalized free product* of the subgroups G_γ. Since S_α and S_β have not been assumed disjoint, we can have

$$G_\alpha \cap G_\beta = H_{\alpha\beta} = H_{\beta\alpha} \neq 1 \ .$$

If $H_{\alpha\beta} = 1$ for all $\alpha \neq \beta$, then G is called the *free product*, or the *ordinary free product* of the subgroups G_γ. In this paper, we assume that all the intersections $H_{\alpha\beta}$ coincide to form a single subgroup H. That is, we assume that $G_\alpha \cap G_\beta = H$, for all $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$. In this case, G is called the *free product of the subgroups G_γ with amalgamated subgroup H*. We denote G by

$$G = \star_{\gamma \in \Gamma}(G_\gamma; H_\gamma) \ ,$$

where H_γ is isomorphic with H for all $\gamma \in \Gamma$. If G is the free product of two subgroups A and B with amalgamated subgroup H, then we write $G = A \star_H B$.

4. **Main Results.** In 1972 C. Y. Tang published the following theorem [11, Lemma 2.5, p. 570]:

Theorem 1. Let $G = A \star_H B$, where H is any cyclic subgroup. If N is any subgroup of H normal in G, then $\Phi(A) \cap N$ and $\Phi(B) \cap N$ are contained in $\Phi(G)$.

Remark. In Theorem 1, since N is any subgroup of H normal in G, clearly, N can be replaced by $K(G, H)$.

Next, we state and prove Theorem 1 for infinitely many free factors.

Theorem 2. Let

$$G = \star_{\gamma\in\Gamma}(G_\gamma; H_\gamma)$$

be the free product of any collection of groups $\{G_\gamma\}_{\gamma\in\Gamma}$ with amalgamated subgroup H, where H is any cyclic subgroup. If N is any subgroup of H normal in G, then $\Phi(G_\gamma)\cap N$ is contained in $\Phi(G)$ for every $\gamma\in\Gamma$.

Proof. Let $x\in\Phi(G_\gamma)\cap N$, where γ is an arbitrary element of Γ. To show that $x\in\Phi(G)$, we need to prove that x is a nongenerator of G. Now, let S be a set consisting of elements of G such that $< S, x > = G$. But, since $< x >$ is a characteristic subgroup of N, and N is a normal subgroup of G, we deduce that $< x >$ is a normal subgroup of G. Hence, for every $g\in G_\gamma$ there exists an element $s_g \in < S >$ such that $g = s_g x^k$, for some integer k. Thus, $s_g = g(x^k)^{-1}\in G_\gamma$. This implies that $G_\gamma = < x, s_g : g\in G_\gamma >$. But, since $x\in\Phi(G_\gamma)$, we have $G_\gamma = < s_g : g\in G_\gamma > \; \leq \; < S >$, and thus $x \in < S >$. Therefore, $G = < S >$. Consequently, $x\in\Phi(G)$. This completes the proof.

Before stating and proving the exact analog of Theorem 2 for the lower near Frattini subgroups, we need the following three well–known theorems. Proofs of these three theorems can be found in [10, pp. 21–26].

Theorem 3. If H and K are any two subgroups of a group G such that $K \subseteq H$, then $|G : K| = |G : H||H : K|$.

Theorem 4. Let H and K be any two subgroups of a group G. If $|G : H|$ is finite, then $|K : H \cap K|$ is finite, and $|K : H \cap K| \leq |G : H|$. Equality holds if and only if $G = HK$.

Theorem 5. Let U and V be any two subsets, and let L be any subgroup of a group G. If $U \subseteq L$, then $UV \cap L = U(V \cap L)$.

Theorem 6. Let

$$G = \star_{\gamma \in \Gamma}(G_\gamma; H_\gamma)$$

be the free product of any collection of groups $\{G_\gamma\}_{\gamma \in \Gamma}$ with amalgamated subgroup H, where H is any cyclic subgroup. If N is any subgroup of H normal in G, then $\lambda(G_\gamma) \cap N$ is contained in $\lambda(G)$ for every $\gamma \in \Gamma$.

Proof. To prove $\lambda(G_\gamma) \cap N$ is contained in $\lambda(G)$, we need to show that every element of $\lambda(G_\gamma) \cap N$ is a non–near generator of G. Let $x \in \lambda(G_\gamma) \cap N$, and suppose that $S \subseteq G$ is such that $|G :< x, S > |$ is finite. We wish to show that $|G :< S > |$ is finite. Now, since $< x >$ is a characteristic subgroup of N, and N is a normal subgroup of G, we deduce that $< x >$ is a normal subgroup of G. Thus,

$$(1) \qquad |G :< x, S > | = |G :< x >< S > |$$

is finite, and by Theorem 4

$$|G \cap G_\gamma :< x >< S > \cap G_\gamma| = |G_\gamma :< x >< S > \cap G_\gamma|$$

is finite. Hence, by Theorem 5 $|G_\gamma :< x > (< S > \cap G_\gamma)|$ is finite. But, since x is a non–near generator of G_γ, and

$$|G_\gamma :< x > (< S > \cap G_\gamma)| = |G_\gamma :< x, < S > \cap G_\gamma > |$$

is finite, we deduce that $|G_\gamma :< S > \cap G_\gamma|$ is finite. Now, intersecting G_γ and $< S > \cap G_\gamma$ with $< x >$, and using Theorem 4 we have

$$(2) \qquad | < x >:< S > \cap < x > | = | < x >< S >:< S > |$$

which is finite. Finally, from (1), (2), and Theorem 3 it follows that

$$|G :< S > | = |G :< x >< S > || < x >< S >:< S > |$$

is finite. Therefore, $\lambda(G_\gamma) \cap N \leq \lambda(G)$. This completes the proof.

5. **Open Questions.**

<u>Question 1</u>. We have used the definition that $\Phi(G)$ is the set of all nongenerators of G, in the proof of Theorem 2. Can Theorem 2 be proven by using the definition that $\Phi(G)$ is the intersection of all maximal subgroups of G?

<u>Question 2</u>. Is the statement of Theorem 6 still true, if $\lambda(G)$ is replaced by $\mu(G)$?

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