FIBONACCI DECIMAL NUMBER PATTERNS
VIA THE GENERATING FUNCTION

Russell Euler

Northwest Missouri State University

The Fibonacci sequence is defined recursively by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. The first few terms of this sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, $\cdots$. To six decimal places, the decimal expansion of 1/89 is .011235. Similarly,

$$\frac{1}{9899} \simeq .0001010203050813213455\,, \quad \text{and}$$

$$\frac{1}{998999} \simeq .000001001002003005008013021034055.$$

Ignoring zeros, the occurrence of Fibonacci numbers in the above decimal expansions is apparent. If the decimal expansion of 1/89 is carried out to one more digit, the digit in the seventh decimal place is 9, not 8. The purpose of this paper is to explain these phenomena.

In [1, p. 3], the generating function for the Fibonacci sequence

$\{F_n\}$ was given to be

$$(1) \qquad \sum_{n=1}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}.$$

The function defined by $f(x) = \frac{x}{1-x-x^2}$ has singularities at $(-1 \pm \sqrt{5})/2$. In magnitude, $(-1 + \sqrt{5})/2$ is the smallest of these two singularities. As a result, the interval of convergence of the series in (1) is $|x| < (\sqrt{5} - 1)/2$. Replacing x with $1/x$ in (1) leads to

$$\sum_{n=1}^{\infty} \frac{F_n}{x^n} = \frac{x}{x^2 - x - 1} ,$$

and the region of convergence will be $|x| > (1 + \sqrt{5})/2$. So, in the region of convergence,

$$\sum_{n=1}^{\infty} \frac{F_n}{x^{n+1}} = \frac{1}{x^2 - x - 1}.$$

Since $x = 10^k$ is within the region of convergence for $k \geq 1$,

$$\left.\frac{1}{x^2 - x - 1}\right|_{10^k} = \sum_{n=1}^{\infty} \frac{F_n}{10^{k(n+1)}}$$

$$(2) \qquad = \frac{1}{10^{2k}} + \frac{1}{10^{3k}} + \frac{2}{10^{4k}} + \frac{3}{10^{5k}} + \frac{5}{10^{6k}} + \cdots.$$

In particular, if $k = 1$, then

$$\frac{1}{89} = \frac{1}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \frac{5}{10^6} + \cdots$$

$$= .01 + .001 + .0002 + .00003 + .000005 + .0000008$$

$$+ .00000013 + \cdots$$

$$\simeq .01123593.$$

Equation (2) explains why the Fibonacci numbers occur in the decimal expansions mentioned earlier — including the spacing of the digits with zeros. Furthermore, it is clear why the digit in the seventh decimal place of 1/89 is 9 instead of 8 — in the addition of the decimals in the terms of the series there is a 'carry over' of a 1 from the next term, 13, of the Fibonacci sequence.

Reference

1. M. K. Azarian, "The Generating Function for the Fibonacci Sequence," *Missouri Journal of Mathematical Sciences*, 2 (1990) 3–4.