ON THE INTERSECTION OF THE SETS OF BASE b SMITH NUMBERS AND NIVEN NUMBERS

Wayne L. McDaniel

University of Missouri–St. Louis

Let $p_1, \ldots, p_r$ be distinct primes and

$$m = \prod_{i=1}^{r} p_i^{e_i}$$

be a base b integer. We denote by $S(b,m)$ the sum of the digits of m, and, by

$$S_p(b,m) = \sum_{i=1}^{r} e_i S(b,p_i) \ ,$$

the sum of all the digits of the prime factors of m. The set

$$\{m : S(b,m) = S_p(b,m), m \text{ composite } \}$$

shall be denoted by S, and the set

$$\{m : S(b,m) | m\} \ ,$$

by N.

During recent years a number of properties of the sets S and N have been established. We list several papers which are relevant in our references, but shall be primarily concerned, in this note, with the fact that each of S and N is an infinite set. Our purpose here is to observe that for $b \geq 8$, S and N have an infinite intersection.

For $b = 10$, the elements of S were named "Smith numbers" by A. Wilansky [8], and Kennedy has named the elements of N "Niven numbers", in honor of I. Niven whose mention of such numbers [9] stimulated interest in their properties and distribution. It is reasonable to refer to the elements of S and N as base b Smith numbers and base b Niven numbers, respectively. Thus, we shall prove the following.

Theorem 1. There exist infinitely many integers which are both base b Smith numbers and base b Niven numbers, for $b \geq 8$.

In [7] (Main Theorem, p. 96), we proved a theorem which has the following result as a special case:

Theorem 2. If, for $b \geq 8$, T is a finite set of base b integers such that $U = \{S_p(b,t) : t \in T\}$ is a complete residue system modulo $S_p(b,b)$, then there exist infinitely many integers n with the property that, for each value of n, there is an element $t \in T$ and a non–negative integer v such that $m = t(b^n - 1)b^v$ is in S.

It was shown ([7], Corollary 1) that a set T satisfying the above condition exists for all $b > 1$. Because the proof of Theorem 2 requires considerable development of preliminary ideas, we will not present the proof here, but will rely heavily upon this result and its proof in proving the following.

Theorem 3. There exist infinitely many triples (n, t, v) such that, for $b \geq 8$, $m = tn(b^n - 1)b^v$ is in S.

Proof. Let T be a finite set of base b integers such that $U = \{S_p(b,t) : t \in T\}$ is a complete residue system modulo $S_p(b,b)$. In the proof of Theorem 2, n was selected subject to two conditions depending only upon the value of b, and the following third condition: n is such that $b^n - 1$ exceeds the maximum element of T. We now let n be an integer satisfying the two conditions depending on b, and let n be such that $(b^n - 1)/n$ exceeds the maximum element of T. (This is obviously possible since there exist infinitely many positive integers satisfying the first two conditions.) Then $b^n - 1$ exceeds the maximum element of the set $\overline{T} = \{tn : t \in T\}$. It is clear that

$$U = \left\{S_p(b,tn) : tn \in \overline{T}\right\} = \left\{S_p(b,t) + S_p(b,n) : t \in T\right\}$$

is a complete residue system modulo $S_p(b,n)$. Hence, by Theorem 2, there exists an element $tn \in \overline{T}$ and a non–negative integer v such that $m = tn(b^n - 1)b^v$ is in S, for $b \geq 8$.

We require the following lemma, also proved in [7] (Lemma 3, p. 95).

Lemma. Let n and k be positive integers such that $k \leq b^n - 1$. If v is a non–negative integer, and $m = k(b^n - 1)b^v$, then $S(b,m) = (b-1)n$.

The proof of Theorem 1 is now immediate.

Proof of Theorem 1. Let $m = tn(b^n - 1)b^v \in$ S. By the above lemma, $S(b,m) = (b-1)n$, and, clearly, $n(b-1)|m$. It follows that $m \in$ N, proving the theorem.

In the case of base 10 integers, a slightly simpler construction is possible: In [4], Kennedy showed that there exist infinitely many integers k such that $10^k \equiv 1 \pmod{9k}$, so that $10^k - 1$ is a Niven number, and in [5] we showed that, for every integer n, there exist integers t and v such that $m = t(10^n - 1) \cdot 10^v$ is a Smith number. If n is chosen such that $10^n - 1$ is a Niven number, then it is clear, by our lemma above, that m is also a Niven number.

Example. Let $m = 14985 = 15(10^3 - 1) = 3^4 \cdot 5 \cdot 37$. The digital sum of both m and the prime factors of m is 27, and 27 divides m.

References

1. C. N. Cooper and R. E. Kennedy, "A Partial Asymptotic Formula for the Niven Numbers," *Fibonacci Quart.*, 26 (1988), 163–168.
2. R. E. Kennedy, "Digital Sums, Niven Numbers, and Natural Density," *Crux Mathematicorum*, 8 (1982), 129–133.
3. R. E. Kennedy and C. N. Cooper, "On the Natural Density of the Niven Numbers," *Coll. Math. J.*, 15 (1984), 309–312.
4. R. E. Kennedy and C. N. Cooper, "Niven Repunits and $10^n \equiv 1 \pmod{n}$," *Fibonacci Quart.*, 27 (1989), 139–143.
5. W. L. McDaniel, "The Existence of Infinitely Many k–Smith Numbers," *Fibonacci Quart.*, 25 (1987), 76–80.
6. W. L. McDaniel and S. Yates, "The Sum of Digits Function and its Application to a Generalization of the Smith Number Problem," *Nieuw Archief Voor Wiskunde*, 7 (1989), 39–51.
7. W. L. McDaniel, "Difference of the Digital Sums of an Integer Base b and Its Prime Factors," *J. of No. Thy.*, 31 (1989), 91–98.
8. A. Wilansky, "Smith Numbers," *Two–Year Coll. Math. J.*, 13 (1982), 21.
9. The 5th Annual Miami University Conference on Number Theory (1977), Oxford, Ohio.