SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the editor.

6. *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Prove

$$\sum_{n \le x} \frac{1}{3n-2} = \frac{1}{3}\log(3x-2) + \frac{1}{6}\log 3 + \frac{\pi}{6\sqrt{3}} + \frac{\gamma}{3} + O\left(\frac{1}{x}\right) ,$$

where log is the natural log and γ is Euler's constant.

Comment by Don Redmond, Southern Illinois University at Carbondale, Carbondale, Illinois.

The problem as given is to find an asymptotic expansion for

$$\sum_{n \le x} \frac{1}{3n-2} .$$

If we look in Ramanujan's notebooks (B.C. Berndt, Ramanujan's Notebooks, part I, p. 185) we find the following in Chapter 8, Entry 7. If x is a positive integer and a and b are arbitrary complex numbers, then

$$\Psi\left(\frac{a}{b} + x + 1\right) - \Psi\left(\frac{a}{b} + 1\right) = b\sum_{k=1}^{x} \frac{1}{a+bk} .$$

Here

$$\Psi(z) = \frac{\Gamma'}{\Gamma}(z) .$$

In Abramowitz and Stegun (Handbook of Mathematical Functions, p. 259) we find an asymptotic expansion for $\Psi(z)$ so that we may generate a general asymptotic expansion for

$$\sum_{n \le x} \frac{1}{a+bn}$$

with a and b complex numbers.

14. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

In triangle ABC, AD is an altitude (with D lying on segment BC). $DE \perp AC$ with E lying on AC. X is a point on segment DE such that $\frac{EX}{XD} = \frac{BD}{DC}$. Prove that $AX \perp BE$.

Solution by Tran van Thuong, Missouri Southern State College, Joplin, Missouri.

From D, we draw $DM \parallel BE$, then we have

$$\frac{BD}{DC} = \frac{EM}{MC} \ . \tag{1}$$

By assumption, we have

$$\frac{BD}{DC} = \frac{XE}{XD} \ . \tag{2}$$

Compare (1) and (2), we see that

$$\frac{EM}{MC} = \frac{XE}{XD} \ .$$

This shows that $XM \parallel DC$. Since $AD \perp BC$, we have $XM \perp AD$. Now looking at $\triangle AMD$, we see that $DE \perp AM$ and $XM \perp AD$; it follows that $AX \perp DM$. Therefore $AX \perp BE$ (since $BE \parallel DM$). This completes the proof.

Also solved by Russell Euler, Northwest Missouri State University, Maryville, Missouri and the proposer.

15. *Proposed by Leonard L. Palmer, Southeast Missouri State University, Cape Girardeau, Missouri.*

Let F_n denote the nth Fibonacci number ($F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-2} + F_{n-1}$ for $n > 2$) and let L_n denote the nth Lucas number ($L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-2} + L_{n-1}$ for $n > 2$). Find all x such that $F_x + L_x \equiv 0 \pmod 4$ and verify.

Composite solution by Bob Prielipp, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin, Russell Euler, Northwest Missouri State University, Maryville, Missouri, and Alex Necochea, University of Texas–Pan American, Edinburg, Texas.

We shall show that $F_x + L_x \equiv 0 \pmod 4$ if and only if $x = 3n - 1$ for some positive integer n. Our solution will use the following known result:

$(*)$ $F_m | F_n$ if and only if $m|n$ for $m \geq 2$.

(For a proof of this theorem, see pp. 334–336 of Burton; Elementary Number Theory (Second Edition); Wm. C. Brown Publishers; Dubuque, Iowa; 1989.) We will also employ the following lemma which is easily proved by using the Binet forms for L_n and F_n.

Lemma. $L_k = F_{k-1} + F_{k+1}$ for each positive integer k.

Now let x be a positive integer. Then from the Lemma,

$$\begin{aligned} F_x + L_x &= F_x + (F_{x-1} + F_{x+1}) \\ &= (F_x + F_{x-1}) + F_{x+1} \\ &= F_{x+1} + F_{x+1} \\ &= 2F_{x+1} \ . \end{aligned}$$

Hence by $(*)$,

$$\begin{aligned} F_x + L_x \equiv 0 \pmod 4 \text{ if and only if } & 2|F_{x+1} \\ \text{iff } & F_3|F_{x+1} \\ \text{iff } & 3|(x+1) \\ \text{iff } & x = 3n-1 \text{ for some positive integer } n \ . \end{aligned}$$

Also solved by J.E. Chance, University of Texas–Pan American, Edinburg, Texas, Bill Wynns and Dale Woods (jointly) Central State University, Edmond, Oklahoma, Donald Skow, University of Texas–Pan American, Edinburg, Texas, and the proposer.

Alex Necochea remarked that an analogous calculation shows that $F_x - L_x = -2F_{x-1}$ for all integers $x \geq 1$, where $F_0 = 0$.

Again, $2 = F_3 | F_{x-1}$ iff $3 | x-1$. Therefore, one has that $F_x - L_x \equiv 0 \pmod 4$ if and only if $x \equiv 1 \pmod 3$. Putting these two results together, one has that

$$F_x \pm L_x \equiv 0 \pmod 4 \text{ iff } x \equiv \mp 1 \pmod 3 .$$

16. *Proposed by Mark Ashbaugh, University of Missouri, Columbia, Missouri.*

Consider the $2n \times 2n$ matrix T_n defined as the skew–symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is –1. For example,

$$T_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$T_2 = \begin{pmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{pmatrix} .$$

Find $\det T_n$ for all positive integers n.

Solution by the proposer.

Let S be the $n \times n$ skew–symmetric matrix with all entries below the main diagonal equal to 1. Then

$$T_n = \begin{pmatrix} S & -I-S \\ I-S & S \end{pmatrix}$$

and thus (adding row 2 to row 1 and then adding column 1 to column 2), we have

$$\begin{aligned} \det T_n &= \det \begin{pmatrix} S & -I-S \\ I-S & S \end{pmatrix} \\ &= \det \begin{pmatrix} I & -I \\ I-S & S \end{pmatrix} \\ &= \det \begin{pmatrix} I & 0 \\ I-S & I \end{pmatrix} \\ &= 1 . \end{aligned}$$

The transformations that we have performed on the partitioned determinants above are easily justified in terms of elementary operations on determinants.

Also solved by Alex Necochea, University of Texas–Pan American, Edinburg, Texas and Sam Cazares and Dale Woods (jointly), Central State University, Edmond, Oklahoma.

Comment by the proposer.

This matrix figured in Problem B–5 in the 1988 Putnam exam. There it sufficed to know that $\det T_n \neq 0$ and this is relatively easy to see by working modulo 2. Here we ask for the explicit evaluation of $\det T_n$.