RESONANCE IN LINEAR DIFFERENTIAL EQUATIONS AND L'HOSPITAL'S RULE*

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Historically, differential equations grew into an independent subject from a branch of calculus. It is very useful to cultivate this connection in both calculus and differential equations courses, all the more that today's changes in calculus teaching will influence our approach to differential equations. From this standpoint, we would like to continue the discussion initiated in [1] on the application of L'Hospital's rule to the study of the solutions of linear differential equations with constant coefficients.

Consider the undamped mass–spring system with a given periodic external force. The equation of motion is

$$x'' + \omega_0^2 x = A \cos \omega t, \tag{1}$$

where ω_0 is the natural frequency of the system and ω is the applied

* Research partially supported by U.S. Army Grant DAAL03–89–G–0107.

frequency. If $\omega \neq \omega_0$, a particular solution

$$x_p(t) = \frac{A}{\omega_0^2 - \omega^2} \cos \omega t \tag{2}$$

can easily be found by the method of undetermined coefficients. If ω is close to ω_0 then x_p represents an oscillation with large amplitude. If we think of ω_0 as fixed and let ω approach ω_0, the amplitude of the oscillation becomes unbounded. This phenomenon is called resonance. If $\omega = \omega_0$, a particular solution cannot be obtained from (2). However, since the numerator in (2) tends to $A \cos \omega_0 t$ as $\omega \to \omega_0$ and

$$x_0(t) = \frac{A}{\omega_0^2 - \omega^2} \cos \omega_0 t$$

is a solution of the equation $x'' + \omega_0^2 x = 0$, it is preferable to take a particular solution of (1) in the form

$$x(t, \omega) = x_p(t) - x_0(t) = \frac{A(\cos \omega t - \cos \omega_0 t)}{\omega_0^2 - \omega^2}. \tag{3}$$

Employing L'Hospital's rule we find the limit of (3) as $\omega \to \omega_0$ which yields a particular solution of (1) in the case $\omega = \omega_0$:

$$\lim_{\omega \to \omega_0} x(t, \omega) = \frac{At}{2\omega_0} \sin \omega_0 t.$$

Incidentally, a similar situation takes place with the formula for the power integral

$$\int x^a dx = \frac{x^{a+1}}{a+1} + C,$$

where $a \neq -1$ is any real number and $x > 0$. Since $x^{a+1} \to 1$ as $a \to -1$, we write

$$C = -\frac{1}{a+1} + C_1$$

and

$$\int x^a dx = \frac{x^{a+1} - 1}{a+1} + C_1.$$

This formula "works" also at $a = -1$, for L'Hospital's rule shows that

$$\lim_{a \to -1} \frac{x^{a+1} - 1}{a+1} = \ln x.$$

Now consider, for example, the equation

$$x'' - 2\omega_0 x' + \omega_0^2 x = Ae^{\omega t}, \tag{4}$$

which admits a particular solution

$$x_p(t) = \frac{A}{(\omega - \omega_0)^2} e^{\omega t}, \omega \neq \omega_0. \tag{5}$$

Again, a particular solution cannot be obtained from here at $\omega = \omega_0$, and in this case it is impossible to alleviate the difficulty by subtracting from (5) the solution

$$\frac{Ae^{\omega_0 t}}{(\omega-\omega_0)^2}$$

of the homogeneous equation corresponding to (4). Indeed, ω_0 is a double root of the denominator $(\omega-\omega_0)^2$ and only a simple root of the numerator $A\big[e^{\omega t}-e^{\omega_0 t}\big]$. To accomplish the task, we expand $e^{\omega t}$ as a function of ω in a Taylor series

$$e^{\omega t}=e^{\omega_0 t}+te^{\omega_0 t}(\omega-\omega_0)+\cdots$$

centered at ω_0 and observe that

$$x_0(t)=\frac{A}{(\omega-\omega_0)^2}\big[e^{\omega_0 t}+(\omega-\omega_0)te^{\omega_0 t}\big]$$

is a solution of the equation $x''-2\omega_0 x'+\omega_0^2 x=0$. Moreover, the difference

$$x(t,\omega)=x_p(t)-x_0(t)=\frac{A}{(\omega-\omega_0)^2}\big[e^{\omega t}-e^{\omega_0 t}-(\omega-\omega_0)te^{\omega_0 t}\big]$$

is a particular solution of equation (4) whose numerator and its derivative with respect to ω approach zero as $\omega\to\omega_0$. By L'Hospital's rule,

$$\lim_{\omega\to\omega_0}x(t,\omega)=\lim_{\omega\to\omega_0}\frac{A\big[te^{\omega t}-te^{\omega_0 t}\big]}{2(\omega-\omega_0)}=\lim_{\omega\to\omega_0}\frac{At^2e^{\omega t}}{2}=\frac{At^2}{2}e^{\omega_0 t},$$

which is a particular solution of (4) when $\omega = \omega_0$. This derivation reveals the nature of the solutions involved and a quick method for their construction. Assuming $\omega \neq \omega_0$, we find by the method of undetermined coefficients solution (5). The sum $S_2(t,\omega)$ of the first two terms in the Taylor series, centered at ω_0, of its numerator serves as the numerator for the particular solution $x_0(t)$ of the homogeneous equation corresponding to (4). Therefore, the numerator in the difference $x_p(t) - x_0(t)$ is the remainder

$$R_2(t,\omega) = Ae^{\omega t} - S_2(t,\omega) = \frac{At^2}{2}e^{\omega_0 t}(\omega - \omega_0)^2 + \cdots$$

of the above Taylor series after the second term, and

$$x(t,\omega) = \frac{R_2(t,\omega)}{(\omega - \omega_0)^2}.$$

Since the first term of $R_2(t,\omega)$ includes the factor $(\omega - \omega_0)^2$ and all other terms include higher powers of $\omega - \omega_0$, we conclude that the desired solution of (4) for $\omega = \omega_0$ is simply the first term of $R_2(t,\omega)$ divided by $(\omega - \omega_0)^2$.

A more general equation

$$L[x] = Ae^{\omega t}, \tag{6}$$

where L is a linear differential operator with constant coefficients, can be treated similarly. Assuming ω is not a characteristic root, that is $L(\omega) \neq 0$, we find by the method of undetermined coefficients a particular solution of (6),

$$x_p(t) = \frac{A}{L(\omega)} e^{\omega t}. \tag{7}$$

If ω_0 is a characteristic root of order m, a particular solution of equation (6) cannot be obtained from (7) when $\omega = \omega_0$. To resolve the difficulty, we expand $Ae^{\omega t}$ in a Taylor series centered at ω_0 and note that the partial sum of the first m terms,

$$S_m(t, \omega) = Ae^{\omega_0 t} \sum_{i=0}^{m-1} \frac{t^i}{i!} (\omega - \omega_0)^i,$$

is a particular solution of the homogeneous equation $L[x] = 0$. If $L(\omega) \neq 0$, then

$$x_0(t) = \frac{S_m(t, \omega)}{L(\omega)}$$

is also a solution of the equation $L[x] = 0$ and

$$x(t, \omega) = x_p(t) - x_0(t) = \frac{R_m(t, \omega)}{L(\omega)} \tag{8}$$

is a particular solution of (6), where

$$R_m(t, \omega) = Ae^{\omega t} - S_m(t, \omega).$$

The limit of (8) as $\omega \to \omega_0$ represents a particular solution of (6) for the case $\omega = \omega_0$. Successively applying L'Hospital's rule to (8) gives

$$\lim_{\omega \to \omega_0} x(t, \omega) = \frac{At^m}{L^{(m)}(\omega_0)} e^{\omega_0 t}. \tag{9}$$

This provides another explanation to the student why a particular solution of (6) must be sought in the form $Bt^m e^{\omega t}$ if ω is a zero of the characteristic polynomial. The reason why solution (8) "works" for all ω also becomes clear from the structure of its numerator and denominator. Indeed, since $R_m(t, \omega)$ is the remainder of the power series for $Ae^{\omega t}$ after the mth term and the polynomial $L(\omega)$ has a root of order m at ω_0, we can write

$$R_m(t, \omega) = \frac{At^m}{m!} e^{\omega_0 t} (\omega - \omega_0)^m + \cdots,$$

$$L(\omega) = \frac{L^{(m)}(\omega_0)}{m!} (\omega - \omega_0)^m + \cdots,$$

where the derivative $L^{(m)}(\omega_0) \neq 0$ and all missing terms contain powers of $\omega - \omega_0$ higher than m. Substituting these expressions in (8) and approaching ω to ω_0 yields (9).

The above results may be viewed as an application to differential equations of the powerful method of differentiation with respect to a parameter, which is important in the theory of improper integrals. This method can also be used to find particular solutions of linear differential equations in the "non–resonance" cases. Assume we want to find a particular solution of the equation

$$x'' + 2x' - 3x = t^2 e^{5t}. \tag{10}$$

The usual way is to apply the method of undetermined coefficients but we can achieve the result also by differentiating twice a particular solution of the equation

$$y'' + 2y' - 3y = e^{\omega t} \tag{11}$$

with respect to the parameter ω. The roots of the corresponding characteristic equation $k^2 + 2k - 3 = 0$ are $k = 1, -3$. Therefore, to avoid the "resonance" case in (11), we take $\omega \neq 1, -3$. A particular solution of (11) is sought in the form $y_p(t, \omega) = Be^{\omega t}$, and substituting it in (11) gives

$$y_p(t, \omega) = \frac{1}{\omega^2 + 2\omega - 3} e^{\omega t}$$

or

$$y_p(t, \omega) = \frac{1}{4}\left[\frac{1}{\omega - 1} - \frac{1}{\omega + 3}\right] e^{\omega t}.$$

Differentiating $y_p(t,\omega)$ twice with respect to ω provides a particular

solution $x_p(t,\omega)$ of the equation

$$x'' + 2x' - 3x = t^2 e^{\omega t},$$

that is,

$$\begin{aligned} x_p(t,\omega) = & \frac{1}{4}\left[\frac{1}{\omega-1} - \frac{1}{\omega+3}\right] t^2 e^{\omega t} \\ & - \frac{1}{2}\left[\frac{1}{(\omega-1)^2} - \frac{1}{(\omega+3)^2}\right] t e^{\omega t} \\ & + \frac{1}{2}\left[\frac{1}{(\omega-1)^3} - \frac{1}{(\omega+3)^3}\right] e^{\omega t}. \end{aligned}$$

To obtain from here a particular solution of (10), it remains to put

$\omega = 5$. Then

$$x_p(t) = \frac{32t^2 - 24t + 7}{1024} e^{5t}.$$

Acknowledgement. I am thankful to the referee for a number of useful suggestions.

Reference

1. R. Euler, "A Note on a Differential Equation," *Missouri Journal of Mathematical Sciences*, Winter (1989), Vol. 1: No. 1, 26–27.