A MATRIX METHOD FOR SOLVING THE POSTAGE STAMPS PROBLEM

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Dedicated to the memory of Professor C. H. Teng

1. **Introduction.** In several recent papers Gilder [1] and Planitz [2] considered the problem of purchasing postage stamps of various denominations so as to meet a fixed budget. If there are n types of stamps this requires the solution of the equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c \tag{1.1}$$

where a_i is the cost of the ith type of stamp, x_i is the number of stamps and c is the budget, where x_i and a_i are non–negative integers. In [1] Gilder discusses the solution in integers of the equation

$$12x_1 + 17x_2 = 100z \ , \tag{1.2}$$

where x_1 is the number of second class stamps (at the old rate of $12p$), x_2 the number of first class stamps (at $17p$), and z the total cost in pounds. Planitz extends the problem by shopping for three types of stamps giving the equation

$$13x_1 + 18x_2 + 22x_3 = c \ . \tag{1.3}$$

Solving (1.2) is a classical problem in diophantine equations provided that the x_i are unrestricted. The novelty of the postage stamp problem lies in the fact that the solution x_i must be non–negative.

To solve (1.2) Gilder [1] and Planitz [2] use the known continued fraction solution for (1.1) for $n = 2$ to generate all integer solutions. The non–negative ones are then obtained from a simple inequality. Planitz reduces the three variable case to a pair of equivalent two variable equations in order to use continued fractions. He again is able to find solutions in terms of inequalities obtained from the parametric representation given by continued fractions. If we purchase a fourth type of stamp and this becomes a problem with four unknowns, this method will not work out directly, because the linear transformation is not easily found.

In this note we extend the approach of Planitz by giving an algorithm which yields all the solutions of (1.1). Our method does not depend on continued fractions, thus it may be applied in direct fashion to equations with any number of variables and may be easily programmed for computer solution.

The proof of the algorithm is given in section 2. Following this we show how Gilder's solution of (1.2) and Planitz's solution of (1.3) may be found from (1.1).

2. **Matrix Solution of (1.1).** In this section we will present an algorithm for finding the general solution of the linear diophantine equation,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{2.1}$$

where a_1, a_2, ... , a_n, b are integers.

Throughout, A_i will denote the ith row vector of matrix A. $\det(A)$ and A^t will denote the determinant of and the transpose of A, respectively. The following two integral row operations are well known:

(1) $E(i,j)$: Interchange the ith and jth row.

(2) $E(i(k),j)$: Add an integral multiple of k times the ith row to the jth row.

If (2.1) has an integral solution, then the greatest common divisor (g.c.d.) of $a_1, a_2, \ldots, a_n$ must divide b. Therefore, without loss of generality we may assume that the g.c.d. of a_1, a_2, ..., a_n is 1.

Let A denote the row matrix of the coefficients of (2.1) and let $X = [x_1, x_2, \ldots, x_n]$. Then equation (2.1) is of the form $AX^t = b$.

The main result of the paper is the following theorem.

<u>Theorem 1</u>. Let

$$C = [A^t, I_n] = \begin{bmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & 0 & \cdots & 1 \end{bmatrix} .$$

We apply integral row operations on C until we have a matrix of the form

$$\begin{bmatrix} 1 & p_{11} & p_{12} & \cdots & p_{1n} \\ 0 & p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} .$$

Let

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} .$$

Then bP_1 is an integral solution of (2.1) and all integral solutions of (2.1) are of the form

$$X = bP_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n$$

for all integers $\lambda_2, \ldots, \lambda_n$.

Proof. Since the g.c.d. of a_1, a_2, ... , a_n is 1, applying the operations $E(i,j)$ and $E(i(k),j)$ on C will give us a matrix of the form

$$\begin{bmatrix} 1 & p_{11} & p_{12} & \cdots & p_{1n} \\ 0 & p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} .$$

Then the matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

is a finite product of elementary matrices.

Therefore every entry p_{ij} of P is an integer and $\det(P) = \pm 1$. Hence P_1, P_2, ..., P_n are linearly independent and form a basis of the real n–dimensional factor space R^n. Since $P_1 \cdot A = 1$ and $P_i \cdot A = 0$ for all $2 \le i \le n$ where dot is the inner product of vectors, we have that bP_1 is an integral solution of (2.1) and every P_i is perpendicular to A for $i \ge 2$. Therefore,

$$X = bP_1 + k_2 P_2 + \cdots + k_n P_n$$

for arbitrary real numbers k_2, ..., k_n is a solution of (2.1).

We now claim that any integral solution of (2.1) must be of the form

$$X = bP_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n$$

where λ_2, ..., λ_n are integers.

Let S be an integral solution of (2.1). Then every component of the vector $S - bP_1$ is an integer and the vector $S - bP_1$ belongs to the subspace of R^n spanned by P_2, ..., P_n. Hence, there are rational numbers y_2, ..., y_n such that

$$S - bP_1 = y_2 P_2 + \cdots + y_n P_n \ .$$

The equation

$$y_2 P_2 + \cdots + y_n P_n = S - bP_1$$

forms a system of linear diophantine equations

$$\begin{aligned} p_{21}y_2 + p_{31}y_3 + \cdots + p_{n1}y_n &= c_1 \\ p_{22}y_2 + p_{32}y_3 + \cdots + p_{n2}y_n &= c_2 \\ \vdots \qquad\qquad & \quad \vdots \\ p_{2n}y_2 + p_{3n}y_3 + \cdots + p_{nn}y_n &= c_n \end{aligned} \tag{2.2}$$

where c_1, c_2, ..., c_n are integers. Therefore, $Y = [y_2, \ldots, y_n]$ is the unique solution of (2.2). Suppose that there is one y_i (say y_2) that is not an integer. Then there are integers k and s with $(k, s) = 1$ and $k \ge 2$ such that $y_2 = \frac{s}{k}$. Hence, there is a prime number p dividing k. Since

$$\pm 1 = \det(P) = \sum_{j=1}^{n} (-1)^{j+1} p_{1j} \det(P_{1j}) \ ,$$

the g.c.d. of $\det(P_{11})$, $\det(P_{12})$, ..., $\det(P_{1n})$ is 1. Then there are integers j_1 and j_2, such that p does not divide $\det(P_{1j_1})$ or $\det(P_{1j_2})$. By Cramer's rule for the system (2.2), we have

$$\frac{d_1}{\det(P_{1j_1})} = y_2 = \frac{s}{k} = \frac{d_2}{\det(P_{1j_2})} ,$$

where d_1 and d_2 are integers. This is impossible. Therefore, y_2, ..., y_n are integers and the theorem is proven.

3. **Postage Stamp Problems.** We now apply the method given in section 2 to solving (1.2) and (1.3). For (1.2)

$$C = \begin{bmatrix} 12 & 1 & 0 \\ 17 & 0 & 1 \end{bmatrix}$$

elementary row (1) and (2) operations gives

$$\begin{bmatrix} 12 & 1 & 0 \\ 17 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 12 & 1 & 0 \\ 5 & -1 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 3 & -2 \\ 5 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & -2 \\ 1 & -7 & 5 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 17 & -12 \\ 1 & -7 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -7 & 5 \\ 0 & 17 & -12 \end{bmatrix} .$$

Thus all solutions of (1.2) are given by

$$[x_1, x_2] = 100z[-7, 5] + \lambda[17, -12] . \tag{3.1}$$

If $z = 2$, the case considered by Planitz [2], $x_1 = -1400 + 17\lambda$ and $x_2 = 1000 - 12\lambda$. Since x_1 and x_2 must be non–negative, $17\lambda \geq 1400$ and $12\lambda \leq 1000$ which implies $\lambda = 83$. Thus $x_1 = 11$ and $x_2 = 4$.

For the extended postage stamp problem described by Eq. (1.3), row reduction on the matrix

$$\begin{bmatrix} 13 & 1 & 0 & 0 \\ 18 & 0 & 1 & 0 \\ 22 & 0 & 0 & 1 \end{bmatrix} \tag{3.2}$$

gives the matrix

$$\begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 14 & -26 & 13 \\ 0 & 4 & -9 & 5 \end{bmatrix} . \tag{3.3}$$

Thus all integral non–negative solutions may be found by solving

$$\begin{aligned} -c + 14\lambda_1 + 4\lambda_2 &\geq 0 \\ 2c - 26\lambda_1 - 9\lambda_2 &\geq 0 \\ -c + 13\lambda_1 + 5\lambda_2 &\geq 0 \ . \end{aligned} \tag{3.4}$$

For example, if $c = 200$, (3.4) has only 3 pairs of solutions, $[\lambda_1, \lambda_2] = [10, 15]$, $[11, 12]$, and $[12, 9]$, and so $[x_1, x_2, x_3] = [0, 5, 5]$, $[2, 6, 3]$, $[4, 7, 1]$.

If we further extend the problem to four types of stamps, this time using United States prices we get the following equation

$$15x_1 + 25x_2 + 39x_3 + 45x_4 = c \ . \tag{3.5}$$

Fifteen cents for postcards, twenty–five cents for letters, thirty– nine cents for air gram and forty–five cents for airmail. We then have the matrix

$$\begin{bmatrix} 15 & 1 & 0 & 0 & 0 \\ 25 & 0 & 1 & 0 & 0 \\ 39 & 0 & 0 & 1 & 0 \\ 45 & 0 & 0 & 0 & 1 \end{bmatrix} . \tag{3.6}$$

This gives

$$
\begin{bmatrix} 1 & 1 & 1 & -1 & 0 \\ 0 & -14 & -15 & 15 & 0 \\ 0 & -11 & -9 & 10 & 0 \\ 0 & -45 & -45 & 45 & 1 \end{bmatrix} \ . \tag{3.7}
$$

For a given c the solution can be found by solving

$$
\begin{aligned}
c - 14\lambda_1 - 11\lambda_2 - 45\lambda_3 &\geq 0 \\
c - 15\lambda_1 - 9\lambda_2 - 45\lambda_3 &\geq 0 \\
-c + 15\lambda_1 + 10\lambda_2 + 45\lambda_3 &\geq 0 \\
\lambda_3 &\geq 0 \ .
\end{aligned} \tag{3.8}
$$

For instance, whenever $c = 200$, (3.8) has only solutions $[\lambda_1, \lambda_2, \lambda_3] = [3, 2, 3]$, $[6, 2, 2]$, $[7, 5, 1]$, $[8, 8, 0]$, $[9, 2, 1]$, $[10, 5, 0]$ and $[12, 2, 0]$, and so the only possible combinations for x_1, x_2, x_3 and x_4 are $[x_1, x_2, x_3, x_4] = [1, 2, 0, 3]$, $[4, 2, 0, 2]$, $[2, 5, 0, 1]$, $[0, 8, 0, 0]$, $[7, 2, 0, 1]$, $[5, 5, 0, 0]$ and $[10, 2, 0, 0]$.

References

1. J. Gilder, "On Buying Postage Stamps," *Math Gazette*, 71 (1987), 110–112.
2. M. Planitz, "A Genralization of the Postage Stamps Problem," *Math Gazette*, 73 (1989), 21–25.