# SEQUENCES, MATHEMATICAL INDUCTION, AND THE HEINE–BOREL THEOREM

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The usual proof of the Heine–Borel Theorem as in [1] and [2], always makes use of the Bolazano–Weierstrass Theorem and/or Cantor's Intersection Theorem. But if the understanding of these theorems exhausts the students' mathematical knowledge, then an understanding of the proof is lost.

The following method of proof, which is simpler but written in the same spirit as Ross's work in [3], uses only results from the theory of sequences and mathematical induction. This not only allows us to avoid the use of these "big" theorems, but also strengthens the students' understanding of sequences and mathematical induction.

Definition. An $n-$dimensional open interval $I$ in $\Re^n$ is defined by the set

$$I = \{(x_1, x_2, \cdots, x_n) | a_i < x_i < b_i; a_i, x_i, b_i \in \Re; 1 \le i \le n\} .$$

Similarly, an $n-$dimensional closed interval $J$ in $\Re^n$ is defined by the set

$$J = \{(x_1, x_2, \cdots, x_n) |\ a_i \le x_i \le b_i; a_i, x_i, b_i \in \Re; 1 \le i \le n\} .$$

Definition. A set $O$ in $\Re^n$ is said to be open if for each $x$ in $O$, there exists an $n$–dimensional open interval $I$ such that $x \in I \subseteq O$. A set $C$ in $\Re^n$ is closed if $\Re^n - C$ is open.

Remark 1. Every $n$–dimensional closed interval in $\Re^n$ is closed.

Definition. A sequence $s$ in a nonempty set $S$ is a function whose domain is the set of natural numbers and whose range is a subset of $S$.

Throughout this paper $N$ is used to denote the set of natural numbers and $R(s)$ is used to denoted the range of a sequence $s$. A sequence $s$ is said to be infinite if $R(s)$ is an infinite set.

Definition. Let $s$ be a sequence in $\Re^n$. A sequence $t$ is called a subsequence of $s$ if there is an increasing function $\phi$ from $N$ to $N$ such that $t = s \circ \phi$.

Definition. A set $B$ in $\Re^n$ is said to be bounded if there exists an $n$–dimensional closed interval $J$ in $\Re^n$ such that $B \subseteq J$. A sequence $s$ in $\Re^n$ is bounded if $R(s)$ is bounded.

Definition. A set $S$ in $\Re^n$ is said to be compact if for every collection $\mathcal{O}$ of open sets in $\Re^n$ such that $S \subseteq \cup\mathcal{O}$, there exists a finite subcollection $\mathcal{O}^*$ of $\mathcal{O}$ such that $S \subseteq \cup\mathcal{O}^*$.

Remark 2. A set $S$ is compact if and only if for every collection $\mathcal{A}$ of $n$–dimensional open intervals with $S \subseteq \cup\mathcal{A}$, there exists a finite subcollection $\mathcal{A}^*$ of $\mathcal{A}$ such that $S \subseteq \cup\mathcal{A}^*$.

3. Every compact set in $\Re^n$ is closed and bounded.

4. Every closed subset of a compact set in $\Re^n$ is compact.

Definition. Let $S$ be a set in $\Re^n$. A point $p$ is a limit point of $S$ if every $n$–dimensional open interval containing $p$ contains a point $q \neq p$ of $S$.

Remark 5. If $S$ is a set in $\Re^n$, then every $n$–dimensional open interval containing p contains infinitely many points of $S$ distinct from $p$.

6. A subset $S$ of $\Re^n$ is closed if and only if $S$ contains all its limit points.

First, we shall show that every sequence in a closed interval in $\Re$ has a convergent subsequence. Note that the following lemma is an immediate result from the definition of limit point.

Lemma 1. Suppose $s$ is a sequence in $\Re^n$, and $l$ is a limit point of $R(s)$. Then $l$ is a limit point of $R(s) - F$ for any finite subset $F$ of $R(s)$.

Lemma 2. Let $s$ be a sequence in $\Re^n$. If $l$ is a limit point of $R(s)$, then some subseqence of $s$ converges to $l$.

Proof. Let $\pi_i$ be the projection map from $\Re^n$ onto its $i$th component. Since

$$\prod_{i=1}^{n}(\pi_i(l) - 1, \pi_i(l) + 1)$$

is an $n$–dimensional open interval containing $l$, there exists a point $s(n_1)$ in

$$(R(s) - \{l\}) \cap \prod_{i=1}^{n}(\pi_i(l) - 1, \pi_i(l) + 1)\ .$$

Assume that there exists a point $s(n_k)$ in

$$(R(s) - \{l, s(n_1), \cdots, s(n_{k-1})\} \cap \prod_{i=1}^{n}\left(\pi_i(l) - \frac{1}{k}, \pi_i(l) + \frac{1}{k}\right)$$

such that $n_1 < n_2 < \cdots < n_k$. Now

$$\prod_{i=1}^{n}\left(\pi_i(l) - \frac{1}{k+1}, \pi_i(l) + \frac{1}{k+1}\right)$$

is an $n$–dimensional open interval containing $I$. By Lemmas 1 and 2, there exists a point $s(n_{k+1})$ in

$$(R(s) - \{l, s(n_1), \cdots, s(n_k)\}) \cap \prod_{i=1}^{n}\left(\pi_i(l) - \frac{1}{k+1}, \pi_i(l) + \frac{1}{k+1}\right)$$

such that $n_{k+1} > n_k$. Define $\phi : N \to N$ by $\phi(k) = n_k$. Then the sequence $t$ defined by $t(k) = s \circ \phi(k)$ is a subseqence of $s$, and it is clearly convergent.

<u>Definition</u>. Let $s$ be a sequence in $\Re$. Then its <u>limit superior</u> is defined by

$$\overline{lim}\ s = \inf_n \sup_{k \geq n} s(k) \ .$$

Similarly, its limit inferior is defined by

$$\underline{lim}\ s = \sup_n \inf_{k \geq n} s(k) \ .$$

<u>Remark 8</u>. Suppose $s$ is a sequence in $\Re$; $u$ and $l$ are real numbers. Then $u = \overline{lim}\ s$ if

(a) given $\epsilon > 0$, there is an $n \in N$ such that $s(k) < u + \epsilon$ for all $k \geq n$, and

(b) given $\epsilon > 0$ and $n \in N$, there is a $k \geq n$ such that $s(k) > u - \epsilon$.

Similarly, $l = \underline{lim}\ s$ if

(a) given $\epsilon > 0$, there is an $n \in N$ such that $s(k) > l - \epsilon$ for all $k \geq n$ and

(b) given $\epsilon > 0$ and $n \in N$, there is a $k \geq n$ such that $s(k) < l + \epsilon$.

9. $\overline{lim}\ s = \underline{lim}\ s = a$ if and only if $\lim\ s = a$.

The following lemma is a special case of an exercise in [4].

Lemma 3. Let $s$ be a bounded infinite sequence in $\Re$. Then $\overline{lim}\ s$ and $\underline{lim}\ s$ are real numbers and they are limit points of $R(s)$.

Proof. $\overline{lim}\ s$ and $\underline{lim}\ s$ are real numbers from the definitions. Let $u = \overline{lim}\ s$ and let $\epsilon$ be an arbitrary positive real number. Then $(u - \epsilon, u + \epsilon)$ is an arbitrary 1−dimensional open interval containing $u$. From Remark 8, $(u - \epsilon, u + \epsilon)$ contains infinitely many elements of $u$. Thus $u$ is a limit point of $R(s)$. Similarly, $\underline{lim}\ s$ is a limit point of $s$.

Lemma 4. Every bounded sequence in $\Re$ has a convergent subsequence.

Proof. Let $s$ be a bounded sequence of real numbers. If $R(s)$ is finite, then there exist $k \in N$ and $a \in R(s)$ such that $s(n) = a$ for $n \geq k$. Thus $s$ is a convergent sequence. If $R(s)$ is infinite, then the result follows from Lemmas 3 and 2.

Theorem 1. Every sequence in $[a, b]$ has a convergent subsequence which converges to a point in $[a, b]$.

Proof. Let $s$ be a sequence in $[a, b]$. Then $s$ is a bounded sequence in $\Re$. According to Lemma 4, $s$ has a convergent subsequence $t$. Let $l = \lim\ t$. If $R(t)$ is finite, then $l$ is a point in $[a, b]$. If $R(t)$ is infinite, then $l$ is a limit point of $R(s)$ and hence a limit point of $[a, b]$. According to Remark 7, $l$ is a point in $[a, b]$.

Secondly, we shall establish the result that every sequence in an $n$–dimensional closed interval has a convergent subsequence by mathematical induction.

Theorem 2. Every sequence in an $n$–dimensional closed interval $J$ has a convergent subsequence in $J$.

Proof. Theorem 1 shows that the result is true when $n = 1$. Assume that the result is true for $n \leq k$. Let $s$ be a sequence in a $(k+1)$–dimensional interval $J$. Since the case $R(s)$ being finite is quite trivial, we assume that $R(s)$ is infinite. For each $i = 1, 2, \ldots, k, k+1$, let $\pi_i$ denote the projection of $J$ onto its $i$th component $[a_i, b_i]$. Let $s^*$ be the sequence defined by

$$s^*(j) = (\pi_1 \circ s(j), \pi_2 \circ s(j), \cdots, \pi_k \circ s(j))$$

for each $j \in N$. Then $s^*$ is a sequence in the $k$–dimensional closed interval

$$\prod_{i=1}^{k} \pi_i(J) \ .$$

By the induction hypothesis, there exists an increasing function $\phi : N \to N$ such that $s^* \circ \phi$ converges to a point $(l_1, \cdots, l_k)$ in

$$\prod_{i=1}^{k} \pi_i(J) \ .$$

Now $\pi_{k+1} \circ s \circ \phi$ is a sequence in $\pi_{k+1}(J)$. Again, by the induction hypothesis, there exists an increasing function $\psi : N \to N$ such that $(\pi_{k+1} \circ s \circ \phi) \circ \psi$ converges to a point $l_{k+1}$ in $\pi_{k+1}(J)$. Without loss of generality, let $\epsilon$ be an arbitrary positive real number such that for each $i = 1, 2, \ldots, k, k+1$,

$$(l_i - \epsilon, l_i + \epsilon) \subset \pi_i(J) \ .$$

There exist $n_1, n_2 \in N$ such that

$$(s^* \circ \phi) \circ \psi(n) \in \prod_{i=1}^{k} (l_i - \epsilon, l_i + \epsilon)$$

for every $n \geq n_1$, and

$$(\pi_{k+1} \circ s \circ \phi) \circ \psi(n) \in (l_{k+1} - \epsilon, l_{k+1} - \epsilon)$$

for every $n \geq n_2$. Let $m = max\{n_1, n_2\}$. Then for each $n \geq m$,

$$(s \circ \phi) \circ \psi(n) = ((s^* \circ \phi) \circ \psi(n)), (\pi_{k+1} \circ \phi) \circ \psi(n)) \in \prod_{i=1}^{k+1} (a_i - \epsilon, a_i + \epsilon) \ .$$

Thus $s \circ \phi \circ \psi$ is a convergent subsequence of $s$ in $J$.

Thirdly, we shall show that every $n$–dimensional closed interval in $\Re^n$ is compact. Note that the following theorem is a special case of Theorem 7.4 in [5].

Theorem 3. Every $n$–dimensional closed interval in $\Re^n$ is compact.

Proof. Let $J$ be an $n$–dimensional interval in $\Re^n$. Suppose for some $\epsilon > 0$, $J$ cannot be covered by finitely many

$$\left\{\prod_{i=1}^{n}(\pi_i(x) - \epsilon, \pi_i(x) + \epsilon) | x \in J\right\} .$$

Let $t_1$ be a point in $J$. Since

$$\prod_{i=1}^{n}(\pi_i(t_1) - \epsilon, \pi_i(t_1) + \epsilon)$$

is not all of $J$, there exists $t_2$ in $J$ such that

$$t_2 \notin \prod_{i=1}^{n}(\pi_i(t_1) - \epsilon, \pi_i(t_1) + \epsilon) .$$

In general, given $t_1, t_2, \ldots, t_k$, let $t_{k+1}$ be a point in $J$ such that

$$t_{k+1} \notin \bigcup_{j=1}^{k}\left[\prod_{i=1}^{n}(\pi(t_j) - \epsilon, \pi_i(t_j) + \epsilon)\right] .$$

Let $s$ be a sequence such that for each $j \in N$, $s(j) = t_j$. Then $s$ is a sequence in $J$ which has no convergent subsequence. This contradicts Theorem 2. Therefore $J$ must be compact by Remark 3.

Finally, we establish our main result, the Heine–Borel Theorem.

Theorem 4 (Heine–Borel). Let $S$ be a set in $\Re^n$. $S$ is compact if and only if $S$ is closed and bounded.

Proof. By Remark 4, it suffices to show that if $S$ is closed and bounded then it is compact. Since $S$ is bounded in $\Re^n$, there exists an $n-$dimensional closed interval $J$ such that $S \subseteq J$. Since $S$ is closed and $J$ is compact, it follows from Remark 5 that $S$ is compact.

## *References*


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