SOME APPLICATIONS OF THE STONE–WEIERSTRASS THEOREM

Tran van Thuong

Missouri Southern State College

Abstract. In this paper, we will apply the Stone–Weierstrass Theorem to study some properties of the spaces of continuous functions on compact spaces. As a consequence, we will be able to construct a decreasing sequence of linear span dense subspaces in $C[0,1]$.

1. **Introduction and Definitions.** Let $C[a,b]$ denote the metric space of all continuous real valued functions on the closed bounded interval $[a,b]$ with the metric d defined by:

$$d(f,g) = ||f-g|| = \max|f(x)-g(x)|$$

for $f,\ g \in C[a,b]$ and $a \leq x \leq b$. Under this norm, $C[a,b]$ is a complete metric space. Similarly, we denote $C(M)$ to be the set of all continuous real valued functions on M, and we define:

$$||f|| = \max|f(x)|$$

for $f \in C(M)$, $x \in M$. If

$$d(f,g) = ||f-g||$$

for $f,\ g \in C(M)$, then d is a metric for $C(M)$. Let E be any set and let F be a family of complex valued functions on E. We say that F is an algebra if it is closed under the operations of addition, multiplication, and multiplication by constants. Furthermore, let G be a subset of F. We say that the family G separates points of E if given $x \neq y$ in E, there is an element $g \in G$ such that $g(x) \neq g(y)$.

In 1885, Weierstrass proved an important approximation theorem as follows:

Weierstrass Approximation Theorem [4]. Let f be any continuous function in $C[a,b]$. For any $\epsilon > 0$, there exists a polynomial P such that

$$|P(x) - f(x)| < \epsilon$$

for $a \leq x \leq b$. Equivalently, we say that the set P of all polynomials is dense in the complete metric space $C[a,b]$.

In 1937, M. H. Stone gave a remarkable generalization of Weierstrass' Theorem. This result applies to any compact space in place of $[a, b]$.

Stone – Weierstrass Theorem [4]. Suppose A is a self–adjoint algebra of complex continuous functions on a compact set K, A separates points on K, and A vanishes at no point of K. Then the uniform closure, B, of A consists of all complex continuous functions on K. In other words, A is dense in $C(K)$.

2. **Some Applications of the Stone–Weierstrass Theorem.**

Theorem 1. Let X be a compact Hausdorff space and let U be a subalgebra of $C(X)$ such that $f \in U$ then $\overline{f} \in U$ ($\overline{f}$ denotes the complex conjugate of the complex valued function f). Also, for any two points x, $y \in X$ with $x \neq y$, there is an element $g \in U$ with $g(x) \neq g(y)$. Then $\overline{U}$ (the uniform closure of U) either coincides with $C(X)$ or else $\overline{U}$ equals $\{f \in C(X) : f(x_0) = 0 \text{ for some } x_0 \in X\}$.

Proof. Case 1: For every $x \in X$, there is some function $f_x \in U$ with $f_x \neq 0$. We are going to show that $\overline{U} = C(X)$. Defining $g_x = f_x \cdot \overline{f}_x \in U$, we have that $g_x > 0$ since $f_x \neq 0$. By the continuity of g_x, we have $g_x > 0$ in some neighborhood V_x of x. It follows that there is an open cover $\{V_a : a \in X\}$ of X such that for each $a \in X$, there is a function $f_a \in U$ such that $f_a > 0$ in V_a. Since X is compact, there exists a finite subcover, say $V_{x_1}, V_{x_2}, \ldots, V_{x_N}$ where $g_{x_i} > 0$ in V_{x_i}. Now, defining

$$G = \sum_{i=1}^{N} g_{x_i} \ ,$$

we have that $G \in U$ and $G > 0$ on X. It follows that $1/G \in C(X)$. Let

$$B = \{c \cdot 1 + f : c \in C \ \text{ and } \ f \in U\} \ .$$

Then B is an algebra, satisfying all the conditions of the Stone–Weierstrass Theorem. It follows that there is a function $h \in B$ such that

$$|1/G - h| < \epsilon$$

where $\epsilon > 0$, or

$$(*) \qquad |1 - Gh| < \epsilon M$$

where $M = \sup\{G\}$ on X. Since $h \in B$, $Gh \in U$, and since ϵ is arbitrary, $(*)$ shows that $1 \in \overline{U}$. This shows that $\overline{U} = C(X)$.

Case 2: There is some $x_0 \in X$ with $f(x_0) = 0$ for all $f \in U$. We will show that

$$\overline{U} = \{f \in C(X) : f(x_0) = 0 \ \text{ for some } \ x_0 \in X\} \ .$$

Let $H \in C(X)$ and $H(x_0) = 0$. Using algebra B introduced in case 1, there exists a function $f \in U$, and a constant c such that $|H - (c + f)| < \epsilon/2$ on X, for any $\epsilon > 0$. Since $H(x_0) = 0 = f(x_0)$, we have $|c| < \epsilon/2$. It follows that $|H - f| < \epsilon$ on X. This shows that

$$\overline{U} = \{f \in C(X) : f(x_0) = 0 \ \text{ for some } \ x_0 \in X\} \ .$$

Theorem 2. For each integer $N \geq 1$, the set of functions

$$\{e^{nx} : n \geq N\}$$

has a linear span dense in $C[0, 1]$.

Proof. First, we put $y = e^x$. By the Weierstrass Approximation Theorem, polynomials in y are dense in $C[1, e]$. Moreover, given any $f \in C[1, e]$ and any $\epsilon > 0$, there is a finite sum

$$A_N y^N + A_{N+1} y^{N+1} + \cdots + A_{N+L} y^{N+L} = Q(y)$$

such that

$$|f(y) - Q(y)| < \epsilon$$

for all $y \in [1, e]$, and for all $N \geq 1$. To see this statement, we apply the Stone–Weierstrass Theorem to the function

$$y^{-N} = 1/y^N \in C[1, e] \ .$$

Also, for any $\delta > 0$, there is a polynomial $P_0(y)$ such that

$$|y^{-N} - P_0(y)| < e^{-2N} \cdot \delta$$

for $1 \leq y \leq e$. Thus there are finite sums $Q_0(y)$, $Q_1(y)$, ..., $Q_{N-1}(y)$, each of the special form

$$(*) \qquad A_N y^N + A_{N+1} y^{N+1} + \cdots + A_{N+L} y^{N+L}$$

such that

$$|1 - Q_0(y)| < e^{-N} \cdot \delta \quad ; \quad 1 \leq y \leq e \ .$$
$$|y - Q_1(y)| < e^{-N+1} \cdot \delta \quad ; \quad 1 \leq y \leq e \ .$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$|y^{N-1} - Q_{N-1}(y)| < e^{-1} \cdot \delta \quad ; \quad 1 \leq y \leq e \ .$$

By the Weierstrass Approximation Theorem, there is a polynomial $P(y)$ such that $|f(y) - P(y)| < \epsilon/2 \quad ; \quad 1 \leq y \leq e$. Now we can approximate terms of degree 0, 1, ... $N-1$ in $P(y)$ by the expressions of the form $(*)$. If $\delta > 0$ is small enough, we can replace $P(y)$ by an expression $Q(y)$ where

$$Q(y) = A_N y^N + A_{N+1} y^{N+1} + \cdots + A_{N+L} y^{N+L}$$

such that

$$(**) \qquad |f(y) - Q(y)| < \epsilon \ .$$

Secondly, given any $g \in C[0,1]$, if we define a function $f \in C[1,e]$ by $f(y) = g(x)$ where $y = e^x$, then $(**)$ becomes

$$|g(x) - Q(y)| < \epsilon \ .$$

This shows that $g(x)$ is approximated to within ϵ by a linear combination of

$$e^{Nx}, e^{(N+1)x}, \ldots, e^{(N+L)x} \quad \text{for } 0 \leq x \leq 1 \ .$$

Corollary. There is a decreasing sequence of linear span dense subspaces in $C[0,1]$.
Proof. Let

$$L_1 = \text{Span}\{e^x, e^{2x}, \ldots\}$$
$$L_2 = \text{Span}\{e^{2x}, e^{3x}, \ldots\}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$L_N = \text{Span}\{e^{Nx}, e^{(N+1)x}, \ldots\}$$

Then $L_1 \supset L_2 \supset \cdots \supset L_N \supset \cdots$. By Theorem 2, L_i is dense in $C[0,1]$ for $i = 1, 2, 3, \ldots$.

Remark. Further applications of the Stone–Weierstrass Theorem can be found in [1], [2], [3], and [4].

References

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