I CAN'T SEE THE MATH FOR THE PROBLEMS

Richard L. Francis

Southeast Missouri State University

The student of mathematics is often called upon to acquire both understanding and computational skill in areas for which he or she has only the slightest notion of underlying purpose. Not always expressed, such frustrations center around the questions, "Why study this topic?" or "What is this topic good for?" In the judgment of many, the obvious standard of importance is that of immediate practicality and routine application in the world of everyday living. If the learner concludes that man cannot build a bridge, erect a skyscraper, or even buy a car with the tools afforded by the various concepts, immediate suspicions generally arise as to topical worth.

Perhaps a basic philosophy is at play here, namely, *mathematics is merely a routine tool*: it is inherently subservient to the other existing disciplines, and is thus not important in its own right. Teacher disposition of the question, "What is this topic good for?" may take undesirable forms as well. Such responses as

"Justification of the topic is beyond the level of this course," or
"You'll learn about that later on"

are of little satisfaction. If the teaching technique is simply that of "getting on with the problems," deep student concern as to topical worth will persist.

The word "problems," at least to many, suggest some of the more unpleasant aspects of mathematics. No doubt, in the sense of routine and non–appealing manipulation, such an impression is easily obtained. Accordingly, and unfortunately, the golden opportunity called "problem solving" often becomes the stumbling block on the path to learning. Is the student thus justified in his thoughts to the effect that he or she cannot see the "math" for the problems? Or is there a point of view that dispels these negative thoughts and wherein problems may be seen in a vastly different light?

The very nature of mathematics strongly suggests that the answer to the latter question is an emphatic YES. Consideration of various of the frequent encounters within this broad area of problems and problem solving lends support to such a position.

One cannot deny the important place of immediate mathematical application. Yet one of the challenges of mathematics is that of presenting a fair and balanced picture as opposed to a distorted or unbalanced one. This point of view encompasses much and leads into many areas of opportunity. It may be the appeal that goes hand–in–hand with intellectual curiosity. Perhaps it is the thrill of exploring the unknown, reaching out into domains of unforeseen application. To some, it may result in regarding mathematics as an art, thus giving the discipline a standard of beauty and elegance all its own. And of course, there is the much desired objective of mathematical appreciation and perspective.

All too often it is because of limited perspective that the student fails in the case for appreciation.

It may well be this last mentioned area of concern, that of appreciation and perspective, wherein the greatest of challenges presents itself. All too often it is because of limited perspective that the student fails in the case for appreciation. The seemingly unrelated components of day–to–day problem encounters are not seen as part of an integrated "something" much bigger than the task at hand would imply. Rather basically, the student is prevented from seeing the forest for the trees. More precisely, he can't see the math for the problems.

Non–appreciation in mathematics, centering around a lack of perspective, may assume any one of many forms. Now listed are some of the more conventional forms in which a limited perspective becomes evident. In particular, the student is often subject to

1. a tendency toward an obsession with detailed mechanical processes, thus not realizing the versatility, diversity, and power of mathematical analysis.
2. a tendency to be unmindful of the difficulties and hard work that preceded the various mathematical achievements over the centuries.
3. a tendency toward a preoccupation with the practical and applied features of mathematics to the place that the deeper meanings of the discipline are slighted.
4. a tendency to compartmentalize, failing thus to see the unity and interrelatedness of mathematics.
5. a tendency toward indifference (or a Ho–Hum attitude) in mathematics, based on a feeling that everything that can be done has been done.

All of the above tendencies have historical overtones.

The Far-Reaching Utility of Mathematics. The first of the listed tendencies manifests itself even in the most fundamental of mathematical forms. Especially noteworthy is the student's frequent encounter with basic symbolism.

Perhaps Archimedes would have discovered the calculus if then he had access to a place–value system of numeration.

Such basic symbolism is obviously an essential in the powerful analytical methods of present–day mathematics. Consider historically, say, Hindu–Arabic numerals and the remarkable scheme whereby they form a numeration system. Computational skills are continually reinforced by use of these symbols in performing the fundamental operations. Excellent contrast, calling attention to the potential of present numeration, is gained by appropriate references to preceding numeration systems. It may be the simple additive–repetitive one of the Egyptians, or, more recently, the alphabetical–numeral system of the early Greeks. Obstacles of great magnitude, thus indicating the mathematical devotion of the ancients, are apparent in such long–ago accomplishments. Imagine their success in establishing the infinitude of the primes or calculating the first four even perfect numbers or discovering the Euclidean algorithm with such tools.

The question is often raised as to what success these long–ago mathematicians might have had if their symbolism had only been better. History, of course, records "what happened" and not "what might have been." Still, conjectures persist. Perhaps Archimedes would have discovered the calculus if then he had access to a place–value system of numeration. Perhaps the mathematically barren era of the Dark Ages would not have been so barren had Roman numeration been replaced earlier by an improved numeration system. Even in more recent times, the mathematician Edward Waring remarked, "Theorems of this kind (Wilson's Theorem) will be very hard to prove because of the absence of a notation to express prime numbers." The dismal forecast, which ends on a bright note, tells us that the story of enhanced symbolism is a continuing one. It brings one forward in time through the syncopated algebra of Diophantus and still later, the highly useful notation of Euler ($y = f(x), e, i$, etc.) and others.

The student who is inclined to regard algebra as a meaningless manipulating of x's and y's would do well to consider what the so–called "meaningless" activity culminates in. Even as the student of music, practicing long and monotonous drill exercises, foresees putting these pieces together in time so as to play the works of Bach and Beethoven, so should the student of mathematics endeavor to see a finished product. The detailed mechanical processes of mathematics, if viewed in a proper historical light, and considered as well in the context of fulfillment, can direct the student's thinking well beyond the routine and lead to a feeling of appreciation.

Taking Mathematics for Granted. A fine historical thought, bringing mathematical appreciation forcefully to mind, is the famous statement of Sir Isaac Newton declaring that if he had seen farther than others, it was because he had stood on the shoulders of giants. Perhaps this remark suggests a twentieth century problem, namely the downward tendency of taking things for granted. More specifically, this is the tendency of area 2 above, and concerns an unawareness of the difficulties and hard work preceding the mathematical achievements of the past.

> *Does the student simply use the table of logarithms, or is he/she inclined to ask from where they came?*

A golden opportunity to rise above this "taking for granted" tendency occurs whenever the student uses varied and lengthy tabular approximations, now so commonplace, but originally very difficult to obtain. Does the learner simply use the table of logarithms, or is he/she inclined to ask from where they came? Were these tables merely given to man on tablets of stone, or did they come about at long last only through much computational labor? Obviously, the student may not understand the rigorous infinite series account underlying the logarithmic origins (or for that matter, of sines, cosines, and tangents). Still a consideration of appropriate historic personalities and their deep desire to further the cause of mathematics is of instructional merit. It creates an awareness that might otherwise be slighted.

The discovery of logarithms constituted an important computational breakthrough in the early years of the post-Renaissance. Moreover, the mathematical community, realizing the potential of the method, was both appreciative of and quick to accept the concept.

Reservations as to the importance of logarithmic techniques were totally lacking. Such historical judgments, based on the perspective of difficult origins, can have a desired effect on the student in the case for appreciation.

The ubiquitous hand calculator of present usage, if not regarded properly, can lead the student even farther away from an appreciation of algorithmic and computational methods. On the other hand, tremendous potential exists for a historical and mathematical consideration that leads to the deeper feelings of appreciation. Consider the demanding fifteen year endeavor of William Shanks in the last century to calculate Pi, by hand, to 707 decimal places. Consider at the same time, say, the calculating of Pi to 2035 decimal places by the ENIAC calculator of Aberdeen, Maryland in 1949, a result obtained in only 70 hours of calculating time. One cannot help but appreciate the timesaving nature of modern calculating devices, and again, the historical origins and mathematical bases of such. Moreover, perspective calls for a proper stress on the priority of things. It raises the question, "Where in the scheme of things is computational method incidental, and where is it paramount?"

The Deeper Meanings of Mathematics. The student may be very quick to ask the why of evaluating Pi to thousands of decimal places. It seems likely that ten decimal places would be adequate for even the most refined of computations. This penetrating question typifies, to some extent, the third of the listed tendencies, that is, a preoccupation with the practical and applied features of mathematics to the place that deeper meanings are obscured.

The mathematical pursuit of the questionably beneficial is often justified by such responses as, in the case for climbing Mount Everest, "because it's there." In pondering such a remark, an awareness is likely created of still another important discipline, that of pure mathematics. Coupled with this is the continuing historical concern of rigorizing the nearly obvious results of previous experimental mathematics. The manipulational question "How?" gives way in some measure to the understanding question "Why?"

> *Earlier, experimental forms of geometry ultimately become formal and demonstrative, and involve such notables as Thales, Pythagoras, and Euclid.*

Looking back, far into the past, traces of "pure and applied" distinctions appear. The extremely practical mathematics of Egyptian and Babylonian culture blends somehow into the pure and rigorous methods of the Greeks. Earlier, experimental forms of geometry ultimately become formal and demonstrative, and involve such notables as Thales, Pythagoras, and Euclid.

Such a concern for the deeper meanings of mathematics, afforded essentially by postulational reasoning, cannot be denied its important place in the realization of a proper perspective, nor its important place in present understanding. This clearer picture of today's mathematics conveys the significance of the meaningful activities called generalization, abstraction, and rigorization.

The modern student of mathematics has an excellent opportunity to enhance the above perspective and thus acquire a deeper mathematical understanding by varied encounters with the essentials of postulational reasoning, namely *undefined terms, definitions, axioms, logic,* *and theorems.* A sharpness of insight and perception is to be gained by consideration of both the shortcomings and achievements of previous mathematicians in this context.

Euclid, rigorous and orderly as he was, made no allowance for undefined terms. Thus he defined the most basic of geometric fundamentals. The perceptive student today realizes the unacceptable consequence of attempting to define everything. It leads either to an unending regression, thus defining a word by use of still more fundamental words (hence an infinite vocabulary) or circularity. Euclid's beginning definition, that of a point, illustrates the matter well. It reads, "a point is that which has no part." Falling short of the goal of defining everything is evident by the question, "What then is a part?" Circularity offers little satisfaction in saying, "A part is that which a point does not have."

Sharpness of insight also relates to a careful consideration of axioms (or assumed statements) and their roles as to consistency and independence. This desired tendency to analyze, not only word meanings, but premises or axioms as well, is illustrated beautifully by the successes and failures of generations of mathematicians. How else could one, in reading Euclid's fifth postulate, properly respond and thus be led to the rationale underlying non–Euclidean geometries? Too, how else could one fully appreciate such "out–of–the–ordinary" thoughts and their tremendous impact in mathematical progress? Here as elsewhere is encountered a removal or setting–aside of traditional confinements, a frequent result of *pure* mathematical pursuit. It allows a perspective of broadened vision and culminates in both logical capability and mathematical appreciation.

Compartmentalization Versus Unity. Success in gaining insight into the deeper meanings of mathematics has considerable bearing on the compartmentalization tendency. The disunity of mathematics, suggested so strongly by traditional approaches, restricts the discipline to little more than a hodge–podge of unrelated subjects. Here perhaps is identified one of the most remarkable features of what is ordinarily called "modern mathematics," namely the crossing of artificial dividing lines among areas of study by virtue of various unifying notions.

> *The perspective of history offers much in the formalizing of unifying principles and their utilization to characterize mathematical thought.*

Great strides were taken in the last century so as to identify similarities and resulting structuring notions. Traces of this appear in such modern day inclusions as the language of sets, postulational methods, formal logic, relations, functions, number systems, and fields. These abstractions, underlying diverse mathematical models, describe a remarkable unity of structure.

Structure is further evident in attempts over the years as to subject matter integration. This is quite evident in the algebraic analysis of geometry by cartesian methods. Even the etymology of the word "geometry" is obscured by such algebraic notions as the "invariants of a figure under a group of transformations." Repeated classroom encounters, much like these, abound in historical overtones. Important names in the historical account range over Cantor, Dedekind, Russell, Klein, Boole, deMorgan, and Hilbert. The perspective of history offers much as to the formalizing of unifying principles and their utilization to characterize mathematical thought.

The student may be inclined to confuse specialization with compartmentalization. Is there basically a difference? Is it inconsistent to speak of the unity of mathematics, and at the same time allow for the specialization in mathematical areas now so evident? No longer, by illustration, can the mathematician say, "I am a geometer." He must also identify the branch of geometry (projective geometry, differential geometry, or whatever), a trend which takes the form of knowing more and more about less and less. Hopefully, a physiological comparison answers the question. The human body, highly diverse in its make-up, possesses an undeniable inter-relatedness or unity or structure. One cannot properly view, say, the skeletal system or the nervous system without considering the body as a whole; this is clearly noncompartmentalization and is desirable. Nevertheless, specialization is the obvious trend in present medical practice; this is also desirable and in no way contradicts the inter-relatedness or unity mentioned above. Only in a similar, specialized way can mathematics be considered in depth.

Misconceptions.

> *The student, in wrongly assessing the remarkable state of mathematical knowledge, may conclude that everything's been done before.*

A curve indicating the pattern of mathematical growth over the various time periods in history would clearly indicate a near knowledge explosion in recent times. Rising slowly, hardly noticeably, in the dim and distant past, such a curve levels off somewhat during the Dark Ages. Leading, however, into the Renaissance, the early modern era, and then the twentieth centry, an impressive state of mathematical achievement is indicated. The student, in wrongly assessing this remarkable state of mathematical knowledge, may conclude that *everything's been done before.* This widespread belief (that the mathematics of today is no longer exciting and adventuresome, having no new worlds to conquer) is the essense of the fifth tendency above.

To counteract such a tendency, a mathematical–historical account of various unsuccessful solving attempts in the area of *famous conjectures* seems promising. Such conjectures as Fermat's Last "Theorem" are simply stated, high in appeal, but quite elusive as to resolution. Tantalizing conjectures, dating from earlier times, continue to baffle present day mathematicians; they create a picture of mathematics which is important, that of a need for further growth, expansion, and development.

Famous conjectures, if used in the suggested instructional manner, can be chosen according to a variety of characteristics. In addition to the "simply–stated feature," such attributes as long–standing, recent, simply named, or inter–disciplinary (geometry, algebra, number theory) may be included. Consider the instructive potential of such conjectures as the infinitude of the set of even perfect numbers or that of prime twins, or the non–existence of odd perfect numbers, or even Goldbach's Conjecture. Consider also the fact that success is not always denied mathematicians in these famous solving endeavors. The proving of the Four Color Map Theorem in 1976 illustrates dramatically that mathematics is neither stagnant nor unchallenging.

Is the subject of mathematics thus a closed book, with no new pages to be written or added? The evidence is overwhelmingly to the contrary.

Established results from relatively recent times convey not only the continual expansion of familiar branches of mathematics but various breakthroughs as well in the case for famous conjectures (as in the Four Color Map Theorem above). Many of these were of historical impact and led directly to branches that were new and vastly different. The Konigsberg Bridge Conjecture, with its tremendous implication in the case for new areas of study, serves as a fitting example. Even more recently would be the setting aside of the conjecture that Euclid's fifth axiom is dependent or provable, thus bringing the mathematical community to the shocking conclusion that non–Euclidean geometries do indeed exist.

Is the subject of mathematics thus a closed book with no new pages to be written or added? The evidence is overwhelmingly to the contrary. Who can say what the late twentieth century counterparts might be to the Konigsberg Bridge Conjecture? In what new and challenging areas might such conjectures culminate? The exciting story is presently unfolding; such an awareness contributes to a vastly clearer perspective.

In no way does the foregoing suggest the unimportance of immediate and routine mathematical performance or application. What it does suggest, however, is that mathematics is more than this. With an appeal to the history of the discipline the point is hopefully made. Moreover, the diligent seeker is consequently assisted in forming meaningful answers to the question, "Why study this topic?" The broadened point of view, so important here, is illustrated well by the example of the two workers, both outwardly doing the same thing, but differing in perspective. One remarks, "I'm chiseling a piece of marble." The other asserts, "I'm building a great cathedral." In a similar way, the student may conclude, "I'm merely factoring the difference of two squares," or he may reason, "I'm learning mathematics." The challenging task is to lead the student to the latter pattern of thought.

References

1. R. L. Francis, "Word Problems: Abundant and Deficient Data," *Mathematics Teacher*, 71 (1978) 6–11.

2. R. L. Francis, "Mathematics by Decree," *Consortium*, Number 23, September 1987.