TRIANGLES WITH EQUIVALENT RELATIONS BETWEEN THE ANGLES AND BETWEEN THE SIDES

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PART I
GEOMETRY

0. Abstract and Introduction.

The simplest examples of equivalent relations between angles and sides for a triangle $\triangle ABC$ with sides a, b, and c are well known, e.g.,

$$\angle A = \angle B \Leftrightarrow a = b \tag{1}$$

and

$$\angle A = \angle B + \angle C \Leftrightarrow a^2 = b^2 + c^2 \ , \tag{2}$$

because the angle–relation is equivalent to $\angle A = \frac{\pi}{2}$.

K. Schwering [7, 8, 9] and J. Heinrichs [3] (see also Dickson [2]) have studied relations of the form $\angle A = n\angle B$ and $\angle A = n\angle B + \angle C$ by the help of trigonometric functions and roots of unity. W. W. Willson [12] and R. S. Luthar [4] have considered the case $n = 2$, and recently J. E. Carroll and K. Yanosko [1] have generalized to the case of n rational. E. A. Maxwell [5, 6] has considered triangles with $2\angle A = \angle B + \angle C$. In this paper we present elementary geometric proofs of the following equivalences:

$$\angle A = 2\angle B \Leftrightarrow a^2 = b^2 + bc \tag{3}$$

$$\angle A = 2\angle B + \angle C \Leftrightarrow a^2 = b^2 + ac \tag{4}$$

$$2\angle A = \angle B + \angle C \Leftrightarrow a^2 = b^2 + c^2 - bc \tag{5}$$

$$\angle A = 2(\angle B + \angle C) \Leftrightarrow a^2 = b^2 + c^2 + bc \tag{6}$$

$$\angle A = 2(\angle B - \angle C) \Leftrightarrow ba^2 = (b - c)(b + c)^2 \tag{7}$$

Furthermore, we present the formulas for the complete set of integral solutions for each of the types of triangles.

1. All Triangles with $\angle A = 2\angle B$.

In $\triangle ABC$ with $\angle A > \angle B$ we draw a line from A to D on a such that $\angle CAD = \angle B$ as in Fig. 1.

Then $\triangle ABC \sim \triangle DAC$, so that

$$\frac{b}{a} = \frac{d}{c} = \frac{a-x}{b} \tag{8}$$

$$\angle A = 2\angle B \Leftrightarrow d = x$$

Figure 1.

or the two equalities

$$ad = bc \tag{9}$$

$$ax = a^2 - b^2 \tag{10}$$

from which we get the valid formula:

$$\frac{x}{d} = \frac{a^2 - b^2}{bc} \ . \tag{11}$$

Hence, we conclude that $\angle A = 2\angle B$ iff $\triangle ADC$ is isosceles or $x = d$, i.e.,

$$\angle A = 2\angle B \Leftrightarrow \angle BAD = \angle B \Leftrightarrow x = d \Leftrightarrow a^2 - b^2 = bc \ . \tag{12}$$

This proves (3).

2. All Triangles with $\angle A = 2\angle B + \angle C$.

In a triangle with $\angle A > \angle B + \angle C$ we draw a line from A to D on a such that $\angle CAD = \angle B$ as in Fig. 2.

Again $\triangle ABC \sim \triangle DAC$ so that we have (10).

Now,

$$\angle A = 2\angle B + \angle C \Leftrightarrow \angle A - \angle B = \angle B + \angle C \ . \tag{13}$$

$$\angle A = 2\angle B + \angle C \Leftrightarrow x = c$$

Figure 2.

But we have that

$$\angle BAD = \angle A - \angle B \tag{14}$$

and

$$\angle BDA = \angle B + \angle C \ . \tag{15}$$

Hence we have

$$\angle A = 2\angle B + \angle C \Leftrightarrow \angle BAD = \angle BDA \Leftrightarrow x = c \Leftrightarrow ac = ax \Leftrightarrow ac = a^2 - b^2 \ . \tag{16}$$

This proves (4).

3. All Triangles with $\angle A = \frac{\pi}{3}$ or $\angle A = \frac{2\pi}{3}$.

The angle relations in (5) and (6) are equivalent to the equations $\angle A = \frac{\pi}{3}$ and $\angle A = \frac{2\pi}{3}$ respectively. The relations of the sides are, of course, just the cosine relations for these angles. But a closer look proves worthwhile.

Suppose $\angle A = \frac{\pi}{3}$ and $\angle C < \angle A < \angle B$. (Signs of equality gives the trivial case of equilateral triangles, $a = b = c$.) Then we draw two equilateral triangles with $\angle A$ of sidelength's respectively c and b as in Fig. 3.

$$\angle A = \angle A' = \frac{\pi}{3} \text{ and } \angle BDC = \frac{2\pi}{3}$$

Figure 3.

Then we notice the pair of solutions $\triangle ABC$ and $\triangle A'BC$ with sides a, b, c and a, b, $b - c$ respectively. Furthermore, $\triangle DBC$ has $\angle BDC = \frac{2\pi}{3}$ and sides a, c, $b - c$.

So, the solutions appear three at a time.

An elementary solution of the cosine relation comes from Pythagoras applied to the triangles $\triangle ABE$ and $\triangle BCE$; i.e.,

$$c^2 - \left(\frac{c}{2}\right)^2 = h^2 = a^2 - \left(b - \frac{c}{2}\right)^2 \tag{17}$$

$\Rightarrow$

$$c^2 = a^2 - b^2 + bc \tag{18}$$

proving (5) from which

$$a^2 = c^2 + (b-c)^2 + c(b-c) \tag{19}$$

proving (6).

4. All Triangles with $\angle A = 2(\angle B - \angle C)$.

This equation is similar to the previous ones so it is surprising that it is equivalent to a third degree equation in the sides; and no longer surprising it is much harder to prove.

We draw in a triangle with $\angle C < \angle B$ the bisector from A to D on a as in Fig. 4.

$$\angle A = 2(\angle B - \angle C) \Leftrightarrow d = c$$

Figure 4.

Then $\angle ADB = \angle C + \frac{1}{2}\angle A$. Hence the relation is equivalent to $\triangle DAB$ being isosceles.

$$\angle A = 2(\angle B - \angle C) \Leftrightarrow \angle C + \frac{1}{2}\angle A = \angle B \Leftrightarrow c = d \ . \tag{20}$$

But this time we need some further lines for support. We extend AB over A to the point E such that $AE = b$, and we extend AD over D to the point F, chosen such that the angle $\angle DCF = \frac{1}{2}\angle A$, as is shown in Fig. 5.

The first extension gives us the isosceles $\triangle CAE$, and hence that $\angle ACE = \angle CEA = \frac{1}{2}\angle CAB$. Therefore

$$\triangle DAB \sim \triangle CEB \ . \tag{21}$$

The second extension gives us

$$\triangle DAB \sim \triangle CAF \sim \triangle DCF \ . \tag{22}$$

The angle relation is thus equivalent to the four similar triangles being simultaneously isosceles.

Proof of the Formula (7):

The four similar triangles and that $AD \parallel EC$ give two equal relations, namely

$$\frac{a-x}{b} = \frac{x}{c} = \frac{a}{b+c} = \frac{z}{y} = \frac{y}{d+z} \tag{23}$$

and

$$\frac{x}{d} = \frac{z}{a-x} \ . \tag{24}$$

Figure 5.

From (23) we get

$$y^2 = z(d+z) = zd + z^2 \ . \tag{25}$$

Using (24) we get

$$y^2 = x(a-x) + z^2 \ . \tag{26}$$

Now using (23) again, this becomes

$$y^2 = \frac{c}{b}(a-x)^2 + y^2\left(\frac{a}{b+c}\right)^2 \tag{27}$$

which can be rewritten as

$$y^2\left(1 - \left(\frac{a}{b+c}\right)^2\right) = \frac{c}{b}(a-x)^2 \ . \tag{28}$$

From this equation we get the equivalence

$$y = a - x \Leftrightarrow \frac{b-c}{b} = \left(\frac{a}{b+c}\right)^2 \ , \tag{29}$$

which completes the proof.

Although this is considerably simpler than our original proof, we suspect a simpler proof can be found. †

5. Geometric Summary.

It is striking that the identical figure arises in cases 1 and 2 and a very similar figure arises in case 5. In cases 1 and 2, we considered two cases of equality between edges of $\triangle ABD$. The trivial case, $c = d$, gives $\angle C = 0$.

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† As a matter of fact, we have received a simpler proof from Professor Andy Liu in Edmonton on July 16, 1991. His proof refers to Figure 4 with the altitude from A added.

Since $\triangle ABC \sim \triangle DAC$, an equality between sides of $\triangle DAC$ makes $\triangle ABC$ isosceles. We can make other simple constructions, e.g. find D so that $\angle ADB = \angle B$; but they do not lead to any other relations.

In case 5, we have the same basic construction, but with AD bisecting $\angle A$. There are six possible equalities of sides in $\triangle ABD$ and $\triangle ACD$. The five other cases do not lead to any new relations.

We have examined all linear relationships $\alpha\angle A + \beta\angle B + \gamma\angle C = 0$ with α, β, γ, $\in \{-2, -1, 0, 1, 2\}$ and all of these reduce to the five cases we have considered. Initially we thought all these cases would lead to second degree relations among the sides, so case 5 and its difficulty were quite unexpected.

PART II

DIOPHANTINE ANALYSIS

6. Integral Solutions.

One of the inspirations for this study is the fact that the 4, 5, 6 triangle has one angle double another [8, 10, 12]. It is very remarkable that the smallest integral solutions of most of our cases have sides which are consecutive integers.

$(2,3,4) = (b,c,a)$ is a solution of (4).

$(3,4,5) = (b,c,a)$ is a right triangle.

$(4,5,6) = (b,c,a)$ is a solution of (3).

$(6,7,8) = (c,a,b)$ is a solution of (7).

$(3,5,7) = (b,c,a)$ is a solution of (6).

$(3,7,8) = (c,a,b)$ is a solution of (5).

$(5,7,8) = (c,a,b)$ is a solution of (5).

We might add $(1,2,3) = (a,b,c)$ is a solution of (3) and it is the smallest integral–sided scalene triangle. An obvious question is: what problem has $(5,6,7)$ as an integral solution?

We also know $(13,14,15)$ is the smallest Heronian triangle with consecutive sides [2, 11], so a general question is: what can be said about the integral–sided triangles $(b-1, b, b+1)$? We have not been able to make any progress on these questions.

7. All Solutions to the Diophantine Equations.

(1). *The complete solution to* $a^2 = b^2 + bc$ *is*

$$(a,b,c) = r(pq, p^2, q^2 - p^2) \tag{30}$$

for (q,p) *coprime,* $2p > q > p$, r *arbitrary.*

Proof: If (a,b,c) are pairwise coprime, (3) is written

$$a^2 = b(b+c) \; . \tag{31}$$

Hence $b = p^2$ and $b + c = q^2$, $q > p$ coprime. The triangular inequality $a + b > c$ gives the condition $q^2 - pq - 2p^2 < 0$ or $q < 2p$.

$q = 3$, $p = 2$ gives $a = 6$, $b = 4$ and $c = 5$.

(2). *The complete solution to* $a^2 = b^2 + ac$ *is*

$$(a,b,c) = r(q^2, pq, q^2 - p^2) \tag{32}$$

for $q > p$ *coprime,* r *arbitrary.*

Proof: If (a,b,c) are pairwise coprime, (4) is written

$$b^2 = a(a-c) \; . \tag{33}$$

Hence $a = p^2$ and $a - c = q^2$, $p > q$ coprime.

$p = 2$, $q = 1$ gives $a = 4$, $b = 2$ and $c = 3$.

(3) AND (4). *The complete solutions to (6) and (5) are respectively*

$$(a,b,c) = \frac{r}{4}(3p^2 + q^2, 4pq, |3p^2 - q^2| - 2pq) \tag{34}$$

$$(a, b, b+c) \quad \text{and} \quad (a, b+c, c), \tag{35}$$

when (p,q) *are coprime odd numbers satisfying* $q \notin [p, 3p]$, *and* r *an arbitrary factor.*

Proof: (5) follows from (19) and (6). To solve (6) we assume (a,b,c) to be pairwise coprime and hence b or c odd. By symmetry we may choose b as odd. Now we write (6) as

$$3b^2 = (2a + 2c + b)(2a - 2c - b) \; . \tag{36}$$

A common prime factor p must be a factor in b and therefore odd. It is also a factor in the sum $4a$ and hence in a, and a factor in the difference $4c + 2b$ and hence in c, a contradiction.

So, the two factors are coprime, and we must have $b = p \cdot q$, such that the following sets are equal:

$$\{2a + 2c + b, 2a - 2c - b\} = \{3p^2, q^2\} \ . \tag{37}$$

By addition we get $4a = 3p^2 + q^2$ and by subtraction we get

$$4c + 2b = |3p^2 - q^2| \tag{38}$$

from which (34) follows.

$p = 3$ and $q = 1$ gives $(a, b, c) = (7, 3, 5)$ etc.

In order to require $c > 0$, we must have either $q < p$ or $3p < q$. Then $p = 1$ and $q = 5$ gives $(a, b, c) = (7, 5, 3)$ etc.

(5). *The complete solution to (7) is*

$$(a, b, c) = r(p(2q^2 - p^2), q^3, q(q^2 - p^2)) \tag{39}$$

for $p < q$ coprime integers and r arbitrary.

Proof: A common factor in a and b will be a factor in c, so we may assume a and b to be coprime.

Let p be a prime factor in b to the power n, i.e.

$$b = p^n \cdot d \ , \tag{40}$$

such that p and d are coprime. Now let

$$c = p^m \cdot f \tag{41}$$

$m \geq 0$ is chosen such that p and f are coprime.

Then we substitute (40) and (41) in (7)

$$p^n \cdot d \cdot a^2 = p^{3m}(p^{n-m}d - f)(p^{n-m}d + f)^2 \ . \tag{42}$$

Hence $n = 3m$. We conclude that there is a q such that $b = q^3$ and a g such that $c = q \cdot g$. Hence we have

$$q^3 a^2 = q^3(q^2 - g)(q^2 + g)^2 \ . \tag{43}$$

Let $h = q^2 - g$, then from

$$a^2 = h(2q^2 - h)^2 \tag{44}$$

we conclude that h is a square, say $h = p^2$.

Then $a = p(2q^2 - p^2)$, $b = q^3$ and $c = q(q^2 - p^2)$. Now $c > 0$ requires $p < q$.

For $p = 1$, $q = 2$ we get $(a, b, c) = (7, 8, 6)$.

8. Right Angled and Isosceles Triangles.

In the cases (5) and (6) we have obviously isosceles solutions and an even equilateral in case (5). In these cases a right angle excludes integral solutions.

If the triangles in the cases (3) or (4) shall be isosceles, the only possibility will be

$$p^2 - q^2 = pq \ , \tag{45}$$

but this equation has no rational solutions because

$$\frac{p}{q} = \frac{1 \pm \sqrt{1+4}}{2} \ . \tag{46}$$

If a triangle in case (7) shall be isosceles, then $\angle A = \angle B$ or $\angle A = \angle C$.

If $\angle A = \angle B$ then $2\angle C = \angle A = \angle B$ and we are in case (3) with $\angle B = \angle C$, proved impossible above.

If $\angle A = \angle C$, then $a = c$ and hence

$$2pq^2 - p^3 = q^3 - qp^2 \tag{47}$$

without integral solutions.

If a triangle in case (3) shall be right angled, then either $\angle B$ or $\angle C$ is right. In the first case the triangle becomes isosceles, in the second a 30°–60°–triangle, neither of these can be integral.

The case (4) can be rewritten as

$$\angle A = \frac{\pi}{2} + \frac{1}{2}\angle B \tag{48}$$

so these triangles are always obtuse–angled.

In the case (7) $\angle C < \angle B$, so only $\angle B$ or $\angle A$ may be right. If $\angle B$ is right, then $\angle A = 0$, so this is not the case. If $\angle A$ is right, then $\angle B = \frac{3\pi}{8}$ and $\angle C = \frac{\pi}{8}$, so one triangle exists.

But the sides must satisfy (7) and Pythagoras. Eliminating a^2 from the equation, we obtain

$$b(b^2 + c^2) = (b - c)(b + c)^2 \ , \tag{49}$$

with the only solution

$$b = (\sqrt{2} + 1)c \ . \tag{50}$$

9. Heronian Triangles.

Some of the triangles turn out to have integral areas. Of course, none of the triangles of (5) or (6) can avoid a factor $\sqrt{3}$ so these are less interesting.

It proves useful to rewrite the parameterizations (30), (32) and (39) as follows.

Now (p, q) are coprime, $p < q$.

	a	b	c
$\angle A = 2\angle B$	pq	p^2	$q^2 - p^2$
$\angle A = 2\angle B + \angle C$	q^2	pq	$q^2 - p^2$
$\angle A = 2(\angle B - \angle C)$	$2pq^2 - q^3$	q^3	$q^3 - qp^2$

These forms make it easy to give a useful table of possible sides:

p	q	p^2	pq	q^2-p^2	q^2	$2pq^2-p^3$	q^3	q^3-qp^2
1	2	$*1$	2	3	4	7	8	6
1	3	$*1$	3	8	9	17	27	24
2	3	4	6	5	9	28	27	15
1	4	$*1$	4	15	16	31	64	60
3	4	9	12	7	16	69	64	28
1	5	$*1$	5	24	25	49	125	120
2	5	$*4$	10	21	25	92	125	105
3	5	9	15	16	25	123	125	80
4	5	16	20	9	25	136	125	45
1	6	$*1$	6	35	36	71	216	210
5	6	25	30	11	36	235	216	66
1	7	$*1$	7	48	49	97	343	336
2	7	$*4$	14	45	49	188	343	315
3	7	$*9$	21	40	49	267	343	252
4	7	16	28	33	49	328	343	231
5	7	25	35	24	49	365	343	168
6	7	36	42	13	49	372	343	91
1	8	$*1$	8	63	64	127	512	504
3	8	$*9$	24	55	64	357	512	440
5	8	25	40	39	64	515	512	312
7	8	49	56	15	64	553	512	120

The $*$ means that a case (3) triangle does not exist because $q \geq 2p$.

Of course, it is not obvious whether any of these have integral area. It is convenient to make use of the formula of Heron,

$$\triangle = \text{Area} = \sqrt{s(s-a)(s-b)(s-c)}\ , \tag{51}$$

where $s = \frac{1}{2}(a+b+c)$.

In the case (3) we obtain

$$
(52)\qquad
\begin{aligned}
s &= \frac{1}{2}(pq + p^2 + q^2 - p^2) = \frac{1}{2}q(p+q) \\
s - a &= \frac{1}{2}qp + \frac{1}{2}q^2 - pq = \frac{1}{2}q(q-p) \\
s - b &= \frac{1}{2}qp + \frac{1}{2}q^2 - p^2 = \frac{1}{2}(q-p)(q+2p) \\
s - c &= \frac{1}{2}qp + \frac{1}{2}q^2 - q^2 + p^2 = \frac{1}{2}(q+p)(2p-q) \ .
\end{aligned}
$$

So, we get

$$
(53)\qquad \triangle^2 = \left(\frac{1}{4}\right)^2 \cdot q^2 \cdot (p+q)^2 \cdot (q-p)^2 \cdot (2p+q)(2p-q) \ .
$$

For $\triangle$ to be an integer, $(2p+q)(2p-q) = 4p^2 - q^2$ must be a square. Considerations (mod 4) show that q must be even, which makes $\triangle^2$ an integer and also makes p odd. Any common factor of $2p+q$ and $2p-q$ must divide their sum $4p$ and their difference $2q$, but $\gcd(4p, 2q)$ can only be 2 or 4. So either

$$
(54)\qquad 2p + q = 4s^2 \quad \wedge \quad 2p - q = 4t^2
$$

or

$$
(55)\qquad 2p + q = 2s^2 \quad \wedge \quad 2p - q = 2t^2 \ .
$$

So, we have two possibilities,

$$
(56)\qquad p = s^2 + t^2 \quad \wedge \quad q = 2(s^2 - t^2)
$$

and

$$
(57)\qquad p = \frac{s^2 + t^2}{2} \quad \wedge \quad q = s^2 - t^2
$$

with (56) to apply for s, t coprime of different parity, and (57) for s, t coprime, both odd.

The area becomes then either

$$\triangle = q \cdot (q^2 - p^2) \cdot s \cdot t \tag{58}$$

or

$$\triangle = \frac{1}{2} \cdot q(q^2 - p^2) \cdot s \cdot t \ , \tag{59}$$

where s, t have different parity in (58) and s, t are both odd in (59).

So, we can make the following table of Heronian triangles:

				a	b	c	$\triangle$
s	t	p	q	pq	p^2	$q^2 - p^2$	$(\frac{1}{2})q \cdot c \cdot s \cdot t$
2	1	5	6	30	25	11	132
3	1	5	8	40	25	39	468
4	1	17	30	510	289	611	73320
5	1	13	24	312	169	407	24420
5	2	29	42	1218	841	923	387660
6	1	37	70	2590	1369	3531	1483020

In the case of (4) we obtain:

$$\begin{aligned} s &= \frac{1}{2}(q^2 + pq + q^2 - p^2) = \frac{1}{2}(q+p)(2q-p) \\ s - a &= q^2 + \frac{1}{2}(pq - p^2) - q^2 = \frac{1}{2}p(q-p) \\ s - b &= q^2 + \frac{1}{2}(pq - p^2) - pq = \frac{1}{2}(q-p)(2q+p) \\ s - c &= q^2 + \frac{1}{2}(pq - p^2) - q^2 + p^2 = \frac{1}{2}p(q+p) \ . \end{aligned} \tag{60}$$

So, we get

$$\triangle^2 = \left(\frac{1}{4}\right)^2 \cdot p^2 \cdot (q^2 - p^2)^2 \cdot (2q - p) \cdot (2q + p) \ . \tag{61}$$

We can use the solutions (56) and (57) with p and q interchanged. We get for the area

(62) $$\triangle = p \cdot (q^2 - p^2) \cdot s \cdot t \quad (s, t \ \text{ even})$$

or

(63) $$\triangle = \frac{1}{2} \cdot p \cdot (q^2 - p^2) \cdot s \cdot t \quad (s, t \ \text{ odd})$$

and the following table:

				a	b	c	$\triangle$
s	t	p	q	q^2	pq	$q^2 - p^2$	$(\frac{1}{2})q \cdot c \cdot s \cdot t$
3	2	10	13	169	130	69	4140
4	3	14	25	625	350	429	72072
5	3	16	17	289	272	33	3960
5	4	18	41	1681	738	1357	488520
6	5	22	61	3721	1342	3237	2136420
7	5	24	37	1369	888	793	333060
7	6	26	85	7225	2210	6549	7151508

In the case of (7) we obtain;

$$s = \frac{1}{2}(2pq^2 - p^3 + q^3 + q^3 - qp^2) = \frac{1}{2}(q+p)(2q^2 - p^2)$$

(64)
$$\begin{aligned} s - a &= pq^2 + q^3 - \frac{1}{2}p^3 - \frac{1}{2}qp^2 - 2pq^2 + p^3 = \frac{1}{2}(q-p)(2q^2 - p^2) \\ s - b &= pq^2 + q^3 - \frac{1}{2}p^3 - \frac{1}{2}qp^2 - q^3 = \frac{1}{2}p(q-p)(2q+p) \\ s - c &= pq^2 + q^3 - \frac{1}{2}p^3 - \frac{1}{2}qp^2 - q^3 + qp^2 = \frac{1}{2}p(q+p)(2q-p)\ . \end{aligned}$$

So, we get

(65) $$\triangle^2 = \left(\frac{1}{4}\right)^2 \cdot p^2 \cdot (q^2 - p^2)^2 \cdot (2q^2 - p^2)^2 \cdot (2q+p) \cdot (2q-p)\ .$$

We can again use the solutions (56) and (57) with p and q interchanged. We get for the area the formulas

$$\triangle = p \cdot (q^2 - p^2)(2q^2 - p^2) \cdot s \cdot t \quad (s, t \ \text{ even}) \tag{66}$$

or

$$\triangle = \frac{1}{2} p(q^2 - p^2)(2q^2 - p^2) \cdot s \cdot t \quad (s, t \ \text{ odd}) \tag{67}$$

and the table:

				a	b	c	$\triangle$
s	t	p	q	$2pq^2 - p^3$	q^3	$q^3 - qp^2$	$\left(\frac{1}{2}\right) \cdot \frac{c}{q} \cdot a \cdot s \cdot t$
3	2	10	13	2380	2197	897	985320
4	3	14	25	14756	15625	10725	75963888
5	3	16	17	5152	4913	561	1275120
5	4	18	41	54684	68921	55637	1484123760
7	5	24	37	51888	50653	29341	720075720

10. Conclusion.

$$d = x \quad \text{or} \quad x = c \quad \text{or} \quad d = c$$

Figure 6.

It is striking that the common feature of the problems investigated are the isosceles triangles involved. Some line segment from A to D on A is drawn, and then either $x = d$, $d = c$, $c = x$ or even $d = c = x$. Because of this similarity, the interpretations as angle relations are very similar, but in spite of this the side relations are different and in particular the case (7) with $d = c$ is surprisingly complicated.

Nevertheless, the question of possible Heronian triangles is answered by the application of the very same Diophantine equation in the three solvable cases,

$$4p^2 - q^2 = r^2 \tag{68}$$

proving a sort of similarity between the side relations too.

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