SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

25. [1990, 140] *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Let P be the free product with amalgamations of any collection $\{C_\gamma\}_{\gamma\in\Gamma}$ of infinite cyclic groups, where Γ is an indexing set of cardinality greater than one.

A. Construct an element $h \in P$ and a subgroup $T < P$ such that $|P : T|$ is infinite but $|P :< h, T > |$ is finite.

B. Is h uniquely determined? Is T uniquely determined?

Solution by the proposer.

Let $C_\gamma =< c_\gamma >$, let $H_\gamma =< c_\gamma^{m_\gamma} >$, where m_γ are natural numbers, greater than one, and let

$$P = \star_{\gamma\in\Gamma}(C_\gamma; H_\gamma)$$

with amalgamated subgroup H.

A. Consider an arbitrary element

$$h = c_\gamma^{qm_\gamma} \quad (\gamma \in \Gamma)$$

of H, where q is a natural number. Now, let

$$T =< c_\gamma^{p_\gamma} : \gamma \in \Gamma > ,$$

where p_γ are prime numbers greater than one and $(p_\gamma, qm_\gamma) = 1$, for all $\gamma \in \Gamma$. We claim that this h is the required element and also the subgroup T defined as above is the required subgroup. (Note that if Γ is finite and p is any prime number greater than one, such that $(p, qm_\gamma) = 1$, then

$$T =< c_\gamma^{p} : \gamma \in \Gamma >) .$$

Proof of claim: $(p_\gamma, qm_\gamma) = 1$ implies that there are integers r_γ and s_γ such that

$$r_\gamma p_\gamma + s_\gamma qm_\gamma = 1 .$$

Thus,

$$c_\gamma = c_\gamma^{r_\gamma p_\gamma + s_\gamma qm_\gamma}$$

is an element of $< T, h >$, so that $< T, h >= P$. Therefore, it only remains to prove that T is of infinite index in P. To establish this fact, we choose α, β, ..., η distinct in Γ, and we show that g_1, g_2, ..., g_r, ... are incongruent $\pmod T$, where

$$g_r = (c_\alpha c_\beta \cdots c_\eta)^r$$

and r is a natural number. For r and s different natural numbers, we wish to verify that

$$g_r^{-1} g_s \notin T \ .$$

If not, then there exists an element $t \in T$ such that

$$g_r^{-1} g_s = (c_\alpha c_\beta \cdots c_\eta)^{s-r} = t = \prod_{i=1}^{n} c_{\gamma_i}^{k_i p_{\gamma_i}} \ ,$$

where k_i is a non–zero integer. Now if $r > s$, then

$$1 = c_\alpha c_\beta \cdots c_\eta \cdots c_\alpha c_\beta \cdots c_\eta t \ .$$

If $\gamma_1 \neq \eta$, no cancellations or simplifications are possible. Also, if $\gamma_1 = \eta$, then

$$c_\eta c_{\gamma_1}^{k_1 p_{\gamma_1}} = c_\eta^{1+k_1 p_{\gamma_1}}$$

is not 1. Thus, we have a non–trivial expression for 1, a contradiction. A similar argument will apply for $r < s$. Therefore, $r \neq s$ implies that

$$g_r \not\equiv g_s \pmod T \ .$$

This completes the proof.

B. Obviously, neither h nor T is unique. Because a different q will produce a different h and also, as primes p_γ change so does T.

26*. [1990, 140] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Prove that

$$\sum_{k=1}^{38} \sin \frac{k^8 \pi}{38} = \sqrt{19} \ .$$

Comment by the editor.

No solutions have been received on this problem to date.

27. [1990, 141] *Proposed by Don Redmond, Southern Illinois University, Carbondale, Illinois.*

Show that

$$\int_1^\infty \frac{t-[t]}{t^2(t+1)^2}(2t+1)dt = \log 2 - \frac{1}{2} \ ,$$

where $\log 2$ denotes the natural logarithm of 2 and $[t]$ is the greatest integer less than or equal to t.

Solution I by Jayanthi Ganapathy, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin.

First note that, for any integer $i \geq 2$, 'integration by parts' yields

$$\begin{aligned}\int_{i-1}^{i} \frac{(t-[t])(2t+1)}{t^2(t+1)^2}dt &= -\frac{t-[t]}{t(t+1)}\bigg|_{i-1}^{i} + \log\left(\frac{t}{t+1}\right)\bigg|_{i-1}^{i} \\ &= -\frac{1}{i(i+1)} + \log\left(\frac{i}{i+1}\right) - \log\left(\frac{i-1}{i}\right) \\ &= -\left(\frac{1}{i} - \frac{1}{i+1}\right) + \log\left(\frac{i}{i+1}\right) - \log\left(\frac{i-1}{i}\right) \ ,\end{aligned}$$

since, for any integer $k \geq 1$, $t-[t] \to 0$ as $t \to k^+$ and $t - [t] \to 1$ as $t \to k^-$. Thus for any integer $n > 1$,

$$\begin{aligned}\int_1^n \frac{(t-[t])(2t+1)}{t^2(t+1)^2}dt &= -\sum_{i=2}^{n}\left(\frac{1}{i} - \frac{1}{i+1}\right) + \sum_{i=2}^{n}\left(\log\left(\frac{i}{i+1}\right) - \log\left(\frac{i-1}{i}\right)\right) \\ &= -\frac{1}{2} + \frac{1}{n+1} + \log\left(\frac{n}{n+1}\right) - \log\left(\frac{1}{2}\right) \ .\end{aligned}$$

Let $n \to \infty$. It now follows that

$$\begin{aligned}\int_1^\infty \frac{(t-[t])(2t+1)}{t^2(t+1)^2}dt &= -\frac{1}{2} - \log\left(\frac{1}{2}\right) \\ &= \log 2 - \frac{1}{2} \ .\end{aligned}$$

Solution II by Kandasamy Muthuvel, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin.

It can be easily shown that

$$\frac{(t-n)(2t+1)}{t^2(t+1)^2} = \frac{1}{t} - \frac{n}{t^2} - \frac{1}{t+1} + \frac{n+1}{(t+1)^2} \ .$$

Hence for every positive integer n,

$$\begin{aligned}
\int_n^{n+1} \frac{t-[t]}{t^2(t+1)^2}(2t+1)dt &= \int_n^{n+1} \frac{(t-n)}{t^2(t+1)^2}(2t+1)dt \\
&= \left(\log t + \frac{n}{t} - \log(t+1) - \frac{n+1}{t+1}\right)\Bigg|_n^{n+1} \\
&= \log(n+1) - \log n - \frac{1}{n+1} - \log(n+2) + \log(n+1) + \frac{1}{n+2} \ .
\end{aligned}$$

Hence the integral in the given problem equals

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{r=1}^{n} \left(\log(r+1) - \log r + \log(r+1) - \log(r+2) + \frac{1}{r+2} - \frac{1}{r+1}\right) \\
&= \lim_{n\to\infty} \left(\log(n+1) + \log 2 - \log(n+2) + \frac{1}{n+2} - \frac{1}{2}\right) \\
&= \lim_{n\to\infty} \left(\log\left(\frac{n+1}{n+2}\right) + \log 2 + \frac{1}{n+2} - \frac{1}{2}\right) \\
&= \log 2 - \frac{1}{2} \ .
\end{aligned}$$

This completes the solution.

Solution III by the proposer.

Let a be a positive real number and b be a nonnegative real number. Then

$$\sum_{k=1}^{\infty} \frac{1}{(ak+b)(a(k+1)+b)} = \frac{1}{a}\sum_{k=1}^{\infty}\left(\frac{1}{ak+b} - \frac{1}{a(k+1)+b}\right) = \frac{1}{a(a+b)} \ . \tag{1}$$

If we write this sum as an integral we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{(ak+b)(a(k+1)+b)} &= \int_{1-}^{\infty} \frac{d[t]}{(at+b)(a(t+1)+b)} \\
&= \int_{1}^{\infty} \frac{dt}{(at+b)(a(t+1)+b)} + \int_{1-}^{\infty} \frac{d([t]-t)}{(at+b)(a(t+1)+b)} \\
&= I_1 + I_2 \ .
\end{aligned} \tag{2}$$

We have

$$\begin{aligned}
I_1 &= \frac{1}{a}\int_1^{\infty}\left(\frac{1}{at+b} - \frac{1}{at+b+a}\right)dt \\
&= \frac{1}{a^2}\left(\lim_{t\to\infty} \log\frac{at+b}{at+b+a} - \log\frac{a+b}{2a+b}\right) \\
&= -\frac{1}{a^2}\log\frac{a+b}{2a+b} \ .
\end{aligned} \tag{3}$$

To deal with I_2 we integrate by parts to get

$$\begin{aligned}
I_2 &= \left.\frac{[t]-t}{(at+b)(at+b+a)}\right|_{1-}^{\infty} - \int_1^{\infty} \frac{(t-[t])}{(at+b)^2(at+a+b)^2}(2a^2t+a^2+2ab)dt \\
&= \frac{1}{(a+b)(2a+b)} - \int_1^{\infty} \frac{(t-[t])(2a^2t+a^2+2ab)}{(at+b)^2(at+a+b)^2}dt \ .
\end{aligned} \tag{4}$$

If we now combine (1)–(4) we obtain

$$\int_1^\infty \frac{(t-[t])(2a^2t+a^2+2ab)}{(at+b)^2(at+a+b)^2}dt = -\frac{1}{a^2(a+b)} - \frac{1}{a^3}\log\frac{a+b}{2a+b} + \frac{1}{a(a+b)(2a+b)}$$

$$= -\frac{1}{a^3}\log\frac{a+b}{2a+b} - \frac{1}{a^2(2a+b)} \ .$$

If we take $a=1$ and $b=0$ we get the desired result.

28. [1990, 141] *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Let ABC be an equilateral triangle with segment lengths as indicated in the diagram. Determine s as a function of a, b and c.

Solution by the proposer.

Using the law of sines in triangle BCO gives

$$\frac{\sin(60^\circ-\phi)}{b} = \frac{\sin(60^\circ-\theta)}{c} \ ,$$

which simplifies to

$$b \sin\theta - c\sin\phi = \sqrt{3}(b\cos\theta - c\cos\phi) \ . \tag{1}$$

However, from triangles ABO and ACO,

$$\cos\theta = \frac{-a^2 + b^2 + s^2}{2bs} \ , \tag{2}$$

and

$$\cos\phi = \frac{-a^2 + c^2 + s^2}{2cs} \ . \tag{3}$$

Substituting (2) and (3) into (1) yields

$$b\sin\theta - c\sin\phi = \frac{\sqrt{3}(b^2 - c^2)}{2s} \ . \tag{4}$$

Since the area of triangle ABC is the sum of the areas of triangles ABO, ACO and BCO,

$$b\sin\theta + b\sin(60^\circ - \theta) + c\sin\phi = \frac{\sqrt{3}s}{2} \ . \tag{5}$$

Adding equations (4) and (5) and simplifying with the use of (2) leads to

$$\sqrt{3}b\sin\theta = \frac{s^2 + a^2 + b^2 - 2c^2}{2s} \ . \tag{6}$$

Now, square equation (6), replace $\sin^2\theta$ with $1 - \cos^2\theta$ and use (2) to arrive at

$$s^4 - (a^2 + b^2 + c^2)s^2 + a^4 + b^4 + c^4 - a^2b^2 - a^2c^2 - b^2c^2 = 0 \ ,$$

which has

$$s = \sqrt{\frac{a^2 + b^2 + c^2 + \sqrt{-3(a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2)}}{2}}$$

as the desired root.