THE ANALYTICAL MACHINERY OF SYMMETRY

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In 1951 one of the greatest mathematicians of this century, Hermann Weyl, delivered a series of remarkable lectures on symmetry at Princeton University. These lectures were designed for a very wide audience: faculty and students, mathematicians and nonmathematicians, people interested in natural sciences, and individuals interested in humanities. Weyl's captivating story, published in 1952 [1], contributed greatly to the understanding of the importance of the mathematical idea of symmetry for both mathematical and physical sciences. In 1967 another interesting book on symmetry appeared [2]. Its authors presented a beautiful and clear exposition of the theory of symmetric polynomials and their applications to such diverse algebraic topics as solving nonlinear equations and systems of equations, proving inequalities and identities, factoring complicated expressions, rationalizing the denominator, etc. Some observations on the use of symmetry in the study of functions have been made in [3]. This study was based on geometric methods and linear transformations. The purpose of the present paper is to survey some applications of symmetry to certain topics of algebra, analytic geometry, and calculus to exhibit the power of symmetry as an important analytical tool.

1. **Solving Systems of Equations.**

Consider first a Chicago All-Star Mathematics Problem: find all ordered pairs (x, y) for which

$$\begin{cases} x^2 + y^2 + x + y = 6 \\ xy + x + y = -1 \ . \end{cases} \tag{1}$$

This is an example of a symmetric system of equations. A system of two equations

$$P(x, y) = 0, \qquad Q(x, y) = 0$$

is called symmetric if $P(x, y)$ and $Q(x, y)$ are symmetric polynomials in two variables, that is,

$$P(x, y) = P(y, x), \qquad Q(x, y) = Q(y, x) \ .$$

Symmetric systems are usually solved by means of the substitutions

$$x + y = s, \qquad xy = p \; . \tag{2}$$

For system (1) this results in

$$x^2 + y^2 = (x+y)^2 - 2xy = s^2 - 2p \; ,$$

and (1) is changed to the form

$$s^2 + s - 2p = 6, \qquad s + p = -1 \; ,$$

whence $s^2 + 3s - 4 = 0$ and $s_1 = 1$, $s_2 = -4$. Therefore, $p_1 = -2$, $p_2 = 3$, which produces two simple systems

$$\begin{cases} x + y = 1 \\ xy = -2 \end{cases} \qquad \text{and} \qquad \begin{cases} x + y = -4 \\ xy = 3 \; . \end{cases}$$

From here, the solutions of the original system are determined: $(-1, 2)$; $(2, -1)$; $(-1, -3)$; $(-3, -1)$.

To find the real solutions of the system

$$\begin{cases} x + y = 1 \\ x^4 + y^4 = 7 \; , \end{cases} \tag{3}$$

substitutions (2) are again utilized, resulting in

$$\begin{aligned} x^4 + y^4 = \left(x^2 + y^2\right)^2 - 2x^2y^2 &= \left(s^2 - 2p\right)^2 - 2p^2 \\ &= s^4 - 4ps^2 + 2p^2 \; . \end{aligned}$$

Hence,

$$s = 1, \qquad 2p^2 - 4s^2p + s^4 = 7,$$

that is,

$$p^2 - 2p - 3 = 0, \quad p_1 = 3, \quad p_2 = -1 \ .$$

This leads to the systems

$$\begin{cases} x + y = 1 \\ xy = 3 \end{cases} \qquad \text{and} \qquad \begin{cases} x + y = 1 \\ xy = -1 \ , \end{cases}$$

the first of which has no real solution. The second system gives the equation $u^2 - u - 1 = 0$, whose solutions are $u = (1 \pm \sqrt{5})/2$. Therefore, system (3) has two real solutions:

$$\begin{array}{l} x = (1 + \sqrt{5})/2 \\ y = (1 - \sqrt{5})/2 \end{array} \qquad \text{and} \qquad \begin{array}{l} x = (1 - \sqrt{5})/2 \\ y = (1 + \sqrt{5})/2 \ . \end{array}$$

2. **Deriving the Quadratic Formula Without Completing the Square.**

The quadratic equation has various solution methods, three of which are explained in typical algebra courses. The simplest of these methods, but one that is not always easily applicable, is by factoring. To factor some complicated trinomials, the student must resort to trial and error. Nonetheless, after solving a sufficient number of equations by factoring, students are at least convinced that the typical quadratic equation has two solutions. Experience shows that this is just the right moment to take the next step, proceeding not to the method of completing the square, but in a quite different direction [4–7]. Namely, let m and n be the solutions of the equation $ax^2 + bx + c = 0$, $a \neq 0$; then we have the following two identities

$$am^2 + bm + c = 0 \quad \text{and} \quad an^2 + bn + c = 0 \ . \tag{4}$$

Subtracting the second from the first gives

$$a\left(m^2 - n^2\right) + b(m - n) = 0 \ ,$$

or

$$(m - n)\left(m + n + \frac{b}{a}\right) = 0 \ .$$

Assume $m \neq n$, then

$$m + n = -\frac{b}{a} \ . \tag{5}$$

Next substitute $b = -a(m+n)$ in one of the identities (4) and obtain $am^2 - am(m+n) + c = 0$, which implies

$$mn = \frac{c}{a} \ . \tag{6}$$

Relations (5) and (6) are the well-known Vieta formulas for the sum and product of solutions of the quadratic equation. The left-hand sides of (5) and (6) are the same symmetric polynomials as those in substitutions (2). Being the simplest of all symmetric polynomials, they are called elementary symmetric polynomials. A remarkable theorem from [2] says that each symmetric polynomial in x and y can be expressed as a polynomial in variables $s = x + y$ and $p = xy$. Francois Vieta (1540–1603) had discovered relationships similar to (5) and (6) between the solutions and coefficients of higher-degree equations. To derive the quadratic formula without completing the square, we square (5), then multiply (6) by 4, and subtract the second result from the first. Hence,

$$(m - n)^2 = \frac{b^2}{a^2} - \frac{4c}{a} = \frac{b^2 - 4ac}{a^2} \ .$$

Taking the square root yields

$$m - n = \pm\sqrt{b^2 - 4ac}/a \ . \tag{7}$$

It remains to solve the system of linear equations (5) and (7), to obtain the formula

$$x = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a \tag{8}$$

for the solutions of the quadratic equation $ax^2 + bx + c = 0$. Note that only one sign need be taken, either positive or negative, in (7) to obtain (8). This helps students to understand why the double sign appears in the quadratic formula. Finally, the Vieta formulas hold true also for $m = n$.

3. **Proving Inequalities.**

Inequalities hold a prominent position in many topics of algebra and calculus, and some important inequalities include only symmetric functions. Note that the inequality $(x-y)^2 \geq 0$ (for all real x and y) can be changed to $x^2+y^2 \geq 2xy$. It is natural to apply here substitutions (2) which proved so successful in solving symmetric systems of equations. Since $x^2+y^2 = s^2-2p$, we get the inequality

$$s^2 \geq 4p \tag{9}$$

which is basic for proving symmetric inequalities in two variables.

To prove the inequality

$$\frac{x^3+y^3}{2} \geq \left(\frac{x+y}{2}\right)^3, \quad \text{for } \ x,y \geq 0$$

we write it in the form

$$\frac{(x+y)\big(x^2-xy+y^2\big)}{2} \geq \left(\frac{x+y}{2}\right)^3$$

and use substitutions (2) to obtain

$$s\big(s^2-3p\big)/2 \geq s^3/8 \ .$$

Since x, $y \geq 0$, this inequality is equivalent to $s^2-3p \geq s^2/4$, that is, to $s^2 \geq 4p$. This proves the original inequality for it is equivalent to (9).

Transforming the inequality

$$8\big(x^4+y^4\big) \geq (x+y)^4$$

by means of (2) gives

$$x^4+y^4 = \big(x^2+y^2\big)^2 - 2x^2y^2 = \big(s^2-2p\big)^2 - 2p^2 = s^4-4s^2p+2p^2$$

and

$$8s^4 - 32s^2p + 16p^2 \geq s^4, \quad 7s^4 - 32s^2p + 16p^2 \geq 0,$$

that is, $\left(s^2 - 4p\right)\left(7s^2 - 4p\right) \geq 0$. Since $s^2 \geq 4p$, then all the more $7s^2 \geq 4p$, which proves the original inequality. In particular, if $x + y = 1$, then $x^4 + y^4 \geq 1/8$.

4. **Deriving the Equations of the Ellipse and Hyperbola.**

Some radical equations can be solved by reduction to systems of equations similar to those in Section 1. For example, to solve the equation

$$x + \sqrt{17 - x^2} + x\sqrt{17 - x^2} = 9 \ ,$$

use $y = \sqrt{17 - x^2}$ and get a symmetric system of equations

$$x + y + xy = 9, \qquad x^2 + y^2 = 17,$$

which is changed by substitutions (2) to the form

$$s + p = 9, \qquad s^2 - 2p = 17 \ .$$

Eliminating p results in $s^2+2s-35 = 0$, $s = 5$, $s = -7$, from which $p = 4$, $p = 16$. According to Vieta's formulas (5) and (6), the system of equations $s = 5$, $p = 4$ is equivalent to the quadratic equation $u^2 - 5u + 4 = 0$, whose solutions are $u = 1$, $u = 4$. Hence, $x = 1$, $y = 4$ or $x = 4$, $y = 1$, that is, the original equation has solutions $x = 1$ and $x = 4$. There are no other real solutions because the system $s = -7$, $p = 16$ is equivalent to the equation $u^2 + 7u + 16 = 0$ which has complex roots.

The above technique of solving radical equations can also be used very effectively in the derivation of the equations of the ellipse and hyperbola [8]. Let $F_1(-c, 0)$ and $F_2(c, 0)$ be the foci of the ellipse centered at the origin, and let $P(x, y)$ be any point on the ellipse. Then by the definition of an ellipse,

$$d_1 + d_2 = 2a \ , \tag{10}$$

where $d_1 = F_1P$ and $d_2 = F_2P$. Using the distance formula,

$$d_1^2 = (x + c)^2 + y^2 \ , \qquad d_2^2 = (x - c)^2 + y^2 \ , \tag{11}$$

results in

$$d_1^2 - d_2^2 = 4cx \ . \tag{12}$$

From (10) and (12), it follows that

$$d_1 - d_2 = 2cx/a \ . \tag{13}$$

Solving (10) and (13) produces the formulas

$$d_1 = a + \frac{c}{a}x \ , \qquad d_2 = a - \frac{c}{a}x \ . \tag{14}$$

These are important in their own right since they give rational expressions for the distances from any point of the ellipse to its foci. Combining (11) and (14), we have

$$(x+c)^2 + y^2 = \left(a + \frac{c}{a}x\right)^2 \ , \qquad (x-c)^2 + y^2 = \left(a - \frac{c}{a}x\right)^2 \ .$$

Simplifying either equation yields

$$\left(1 - \frac{c^2}{a^2}\right)x^2 + y^2 = a^2 - c^2$$

and substituting $a^2 - c^2 = b^2$, we obtain the equation of the ellipse $x^2/a^2 + y^2/b^2 = 1$. In a similar manner, the equation of the hyperbola can be derived.

5. **Miscellaneous Examples.**

Even functions (symmetric about the y-axis) and odd functions (symmetric about the origin) are important in algebra and calculus. Thus, for any even function $f(x)$ we have

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx \ ,$$

and for any odd function $g(x)$,

$$\int_{-a}^{a} g(x)dx = 0 \ .$$

The derivative (if it exists) of an even function is odd, and the derivative of an odd function is even. Indeed, every even function satisfies $f(-x) = f(x)$, and differentiating this identity gives $-f'(-x) = f'(x)$, which implies that $f'(x)$ is an odd function. For an odd function we have $g(-x) = -g(x)$, hence $g'(-x) = g'(x)$, which means that $g'(x)$ is even. Some other types of symmetry arise in geometry, algebra, and calculus. In particular, "perimeter = area" problems lead to an interesting class of functions whose graphs are symmetric about the line $y = x$. These functions coincide with their own inverses, $f(x) = f^{-1}(x)$, and are called involutions [9, 10]. Examples of involutions include $y = x$, $y = a - x$, $y = 1/x$, $y = \sqrt[3]{a - x^3}$. Consider two integrals

$$C = \int_0^{\frac{\pi}{2}} \cos^2 x dx \ , \qquad S = \int_0^{\frac{\pi}{2}} \sin^2 x dx \ .$$

Integrating the identity $\cos^2 x + \sin^2 x = 1$ between 0 and $\pi/2$ yields $C + S = \pi/2$. Furthermore, the substitution $y = (\pi/2) - x$ is an involution, which maps the interval $[0, \pi/2]$ onto itself and transforms either integral to the other. Hence, $C = S = \pi/4$.

References

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