ON THE STRUCTURE OF THE HOPF REPRESENTATION RING OF THE SYMMETRIC GROUPS

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The main purpose of this paper is to prove a structure theorem of the graded Hopf representation ring of the symmetric groups $R(S)$. We establish a Hopf ring isomorphism between $R(S)$ and the graded polynomial Hopf ring in an infinite number of variables

$$C = Z[y_1, y_2, \ldots, y_k, \ldots] \ ,$$

by using the λ–operations in $R(S)$ given in a previous paper [8] in terms of outer plethysms.

1. Introduction. In [8] λ–operations are introduced in the graded Hopf representation ring of the symmetric groups

$$R(S) = \{R(S_n) : n \geq 0\}$$

in terms of outer plethysms and it has been shown that with respect to these operations $R(S)$ is a special λ–ring. Zelevinsky [9] developed a complete structure theory of Hopf algebras satisfying the positivity and self adjointness, which is similar to the classical theory of Hopf algebras with commutative multiplication and comultiplication over a field of characteristic zero [7]. In this Hopf algebra approach, Zelevinsky showed $R(S)$ is isomorphic to the polynomial Hopf ring

$$C = Z[y_1, y_2, \ldots, y_k, \ldots]$$

in an infinite number of variables over the integers. Following an elegant proof of Liulevicius [5] we are going to reproduce the Zelevinsky structure theorem in the context of the λ–ring structure in $R(S)$.

Let S_n be the symmetric group of degree n. Let $R(S_n)$ denote the Grothendieck representation group of S_n, then we have a graded group

$$R(S) = \{R(S_n) : n \geq 0\}$$

by setting

$$(R(S))_{2n} = R(S_n) \ ,$$

and

$$(R(S))_{2n+1} = 0 \ ,$$

for all non–negative integers n; where

$$R(S_0) = Z \ .$$

A multiplication in $R(S)$ is the map

$$m_{p,q} : R(S_p) \otimes R(S_q) \to R(S_{p+q}) \ ,$$

defined by

$$m_{p,q} = \mathrm{Ind}_{S_p \times S_q}^{S_{p+q}} \circ \Psi_{p,q} \ ,$$

where

$$\Psi_{p,q} : R(S_p) \otimes R(S_q) \to R(S_p \times S_q)$$

is the canonical isomorphism, and

$$\mathrm{Ind}_{S_p \times S_q}^{S_{p+q}} : R(S_p \times S_q) \to R(S_{p+q}) \ ,$$

the map induced by the embedding of $S_p \times S_q$ as a subgroup of S_{p+q}. A comultiplication on $R(S)$ is the map

$$\triangle_n : R(S_n) \to \sum_{p+q=n} R(S_p) \otimes R(S_q) \ ,$$

defined by

$$\triangle_n = \sum_{p+q=n} \Psi_{p,q}^{-1} \circ \mathrm{Res}_{S_p \times S_q}^{S_{p+q}} \ .$$

In [6] and [9] the following result was proved:

Theorem 1.1. $R(S)$ is a graded Hopf ring with respect to the multiplication $m_{p,q}$ and the comultiplication $\triangle_n$.

2. Outer Plethysms and λ–Rings. Let S_n and S_k be symmetric groups of degree n and k respectively. The wreath product $[S_n]S_k$ of S_n by S_k (the usual notation for $[S_n]S_k$ is $S_n \wr S_k$) is the set $S_n^k \times S_k$ with a multiplication defined by

$$(a_1, a_2, \ldots, a_k; \sigma)(b_1, b_2, \ldots, b_k; \tau) = (a_1 b_{\sigma^{-1}(1)}, \ldots, a_k b_{\sigma^{-1}(k)}; \sigma\tau) \ ,$$

where $a_i,\ b_i \in S_n$ for $k \geq i \geq 1$ and $\sigma,\ \tau \in S_k$. Clearly under this multiplication, $[S_n]S_k$ is a group. Let M be an S_n–module and N be an S_k–module. Then $M^{\otimes k} \otimes N$ is an $[S_n]S_k$–module where the group action is given by:

$$(a_1, \ldots, a_k; \sigma)(m_1 \otimes \cdots \otimes m_k \otimes n) = (a_1 m_{\sigma^{-1}(1)} \otimes \cdots \otimes a_k m_{\sigma^{-1}(k)} \otimes \sigma n\ ),$$

where $a_i \in S_n$, $\sigma \in S_k$, $m_i \in M$ and $n \in N$. In what follows $\otimes$ means $\otimes_C$ and we interpret $M^{\otimes 0}$ as the 1–dimensional S_n–module C, on which S_n acts trivially. The map $\beta : [S_n]S_k \to S_{kn}$ given by

$$\beta((a_1, a_2, \ldots, a_k; \sigma)) = \begin{pmatrix} (j-1)n + i \\ (\sigma(j)-1)n + a_{\sigma(j)}(i) \end{pmatrix} ,$$

$n \geq i \geq 1$, $k \geq j \geq 1$, is a canonical embedding of $[S_n]S_k$ into S_{kn}. Thus we have the induction homomorphism

$$\beta_! = \mathrm{Ind}_{[S_n]S_k}^{S_{kn}} : R([S_n]S_k) \to R(S_{kn}) \ .$$

Definition. The outer plethysm on $R(S)$ is a map $\varphi_{[M]} : R(S) \to R(S)$ given by

$$\varphi_{[M]}([N]) = [\mathrm{Ind}_{[S_n]S_k}^{S_{kn}} (M^{\otimes k} \otimes N)] \ ,$$

where M is an S_n–module and N is an S_k–module.

In James–Kerber's notation [3], the outer plethysm $\varphi_{[M]}([N])$ is denoted by $[M] \odot [N]$. In [8], this outer plethysm is used to construct λ–operations on $R(S)$, and it is shown that with respect to these operations $R(S)$ is a special λ–ring (see [1, 2, 4] for definitions and basic results about λ–rings). The λ–operations $\lambda : R(S) \to R(S)$ are defined for each non–negative integer k by

$$\lambda^k([M]) = [M] \odot [\eta_k] \ ,$$

where $[M] \in R(S_n)$ and $[\eta_k] \in R(S_k)$, is the trivial sign representation of S_k.

Zelevinsky [9] has shown that $R(S)$ is a Hopf ring, which is isomorphic to the polynomial Hopf ring

$$C = Z[y_1, y_2, \ldots, y_k, \ldots]$$

in an infinite number of variables. Our goal in this paper is to establish a Hopf ring isomorphism between $R(S)$ and C in terms of the λ–operations in $R(S)$.

3. The Classical Hopf Algebra C. Let $Z[t]$ be the graded polynomial algebra on one indeterminate t, where the grade of t is 2. Let $\Gamma = Z[t]^*$ be the graded dual of $Z[t]$ with generators y_k defined by $y_k(t^k) = 1$. If $\triangle$ is the comultiplication, then

$$\triangle(y_k) = \sum_{i+j=k} y_i \otimes y_j \ .$$

Let

$$C = Z[y_1, y_2, \ldots, y_k, \ldots] \ ,$$

with $y_0 = 1$, and define a Hopf ring structure on C by making the inclusion $i : \Gamma \to C$ a homomorphism of corings.

The map i has the following universal property: if A is a graded associative algebra and $\varphi : \Gamma \to A$ is a homomorphism of graded abelian groups then there exists a unique homomorphism of graded algebras $\overline{\varphi} : C \to A$ such that the following diagram commutes

$$\begin{array}{ccc} C & & \\ \uparrow{\scriptstyle i} & & \\ \Gamma & \xrightarrow[\varphi]{} & A \ . \end{array}$$

Moreover if A is a Hopf algebra and φ is a map of coalgebras then $\overline{\varphi}$ is a Hopf algebra homomorphism.

By taking the graded dual over Z we obtain a ring homomorphism $i^* : C^* \to Z[t]$, such that for any partition

$$\pi = \{1^{\pi_1}, 2^{\pi_2}, \ldots, k^{\pi_k}\}$$

of k (in notation, $\pi \vdash k$) we have

$$i^*((y_\pi)^*) = \begin{cases} t^k, & \text{if } \pi = \{k\}; \\ 0, & \text{otherwise,} \end{cases}$$

where

$$y_\pi = \prod_{i=1}^{k} y_{\pi_i} \ .$$

The map i^* has the following universal property: if B is a graded coalgebra with each B_n a free abelian group of finite rank, then given a homomorphism of graded abelian groups $\Theta : B \to Z[t]$ there exists a unique coalgebra homomorphism $\overline{\Theta} : B \to C^*$ such that the following diagram commutes

$$\begin{tikzcd} C^* \arrow[d, "i^*"] & \\ Z[t] & B \arrow[l, "\Theta"'] \end{tikzcd} \ .$$

Moreover if B is a Hopf algebra and Θ is an algebra homomorphism then $\overline{\Theta}$ is a Hopf algebra homomorphism.

Define a ring homomorphism $h : \Gamma \to Z[t]$ by setting $h(y_n) = t^n$. Then extend this to an algebra homomorphism which we also denote by h, $h : C \to Z[t]$.

Recall that to determine if a Hopf algebra homomorphism is a monomorphism it is sufficient to show that it is a monomorphism on the primitive elements, these are the elements x of a Hopf algebra with the property

$$\triangle(x) = x \otimes 1 + 1 \otimes x \ ,$$

where $\triangle$ is the comultiplication. The primitives of C have been studied by Newton (cf. [5]). The following is Theorem C of [5].

<u>Theorem 3.1</u>. If R is a commutative ring with unit, then the primitives of $R \otimes C_{2n}$ are all of the form $r \otimes p_n$ where $r \in R$ and $p_n \in C_{2n}$ and they satisfy the following recursion relation:

$$p_n - y_1 p_{n-1} + y_2 p_{n-2} + \cdots + (-1)^{n-1} y_{n-1} p_1 + (-1)^n n y_n = 0 \ .$$

Lemma 3.2. For each n, $h(p_n) = (-1)^{n+1}t^n$.

Proof. It follows immediately from Newton's recursion relation.

Now let $h^* : \Gamma \to C^*$ be the graded dual of the map $h : C \to Z[t]$. Note that $i^* \circ h^* = h$, and since h is an algebra homomorphism, h^* is a coalgebra homomorphism, hence its extension $h^* : C \to C^*$ is a Hopf algebra homomorphism satisfying for each integer $k \geq 1$ the following relation

$$h^*(y_k) = \sum_{\pi \vdash k} (y_\pi)^* \ .$$

Theorem 3.3. The map $h^* : C \to C^*$ is a Hopf algebra isomorphism.

Proof. First note that $i^* \circ h^* = h$, and since $h(p_n) = (-1)^{n+1}t^n$, then h^* is a monomorphism of Hopf algebras. But since C and C^* are free abelian groups of the same rank in each grading, then h^* is an isomorphism.

4. A Hopf Ring Structure of $R(S)$. In this section we are going to show how to use the λ–operations of $R(S)$ to establish a Hopf algebra isomorphism between the graded Hopf representation ring $R(S)$ and the Hopf ring

$$C = Z[y_1, y_2, \ldots, y_k, \ldots] \ .$$

First recall that the only primitive irreducible representation in $R(S)$ is the trivial 1–dimensional representation of S_1, namely $\rho_1 = [1_{S_1}]$. Define a map $\Lambda : C \to R(S)$, by

$$\Lambda(y_k) = \lambda^k(1_{S_1}) = [1_{S_1}] \odot [\eta_k] \ .$$

Theorem 4.1. The map $\Lambda : C \to R(S)$ is a Hopf ring homomorphism.

Proof. It is routine to verify that Λ is a ring homomorphism. Thus it remains to show

Λ commutes with the comultiplication. Consider

$$\begin{aligned}
\triangle_k(\Lambda(y_k)) &= \triangle_k([1_{S_1}] \odot [\eta_k]) \\
&= \triangle_k([\mathrm{Ind}_{[S_1]S_k}^{S_k} 1_{S_1}^{\otimes k} \otimes \eta_k]) \\
&= \triangle_k([1_{S_1}^{\otimes k} \otimes \eta_k]) \\
&= \sum_{p+q=k} (\Psi_{p,q}^{-1} \circ \mathrm{Res}_{S_p \times S_q}^{S_{p+q}})([1_{S_1}^{\otimes k} \otimes \eta_k]) \\
&= \sum_{p+q=k} \Psi_{p,q}^{-1}([\mathrm{Res}_{S_p \times S_q}^{S_{p+q}}(1_{S_1}^{\otimes k} \otimes \eta_k)]) \\
&= \sum_{p+q=k} \Psi_{p,q}^{-1}([(1_{S_1}^{\otimes p} \otimes \eta_p) \otimes (1_{S_1}^{\otimes q} \otimes \eta_q)]) \\
&= \sum_{p+q=k} [(1_{S_1}^{\otimes p} \otimes \eta_p)] \otimes [(1_{S_1}^{\otimes q} \otimes \eta_q)] \\
&= \sum_{p+q=k} \Lambda(y_p) \otimes \Lambda(y_q) \\
&= (\Lambda \otimes \Lambda)(\triangle_k(y_k)) \ .
\end{aligned}$$

Hence, Λ is a homomorphism of Hopf rings.

To prove that Λ is a Hopf ring isomorphism we need the following

Lemma 4.2. The map $\Phi : R(S) \to C^*$ defined by

$$\Phi([M])(y) =< [M], \Lambda(y) > \ ,$$

where $[M] \in R(S)$, $y \in C$, and $< \cdot, \cdot >$ denotes the Schur inner product in $R(S)$, is a ring homomorphism.

Proof. Recall that the multiplication and comultiplication in $R(S)$ are dual under the Schur inner product. If $[M] \in R(S_n)$ and $[N] \in R(S_m)$ and $i : S_n \times S_m \to S_{n+m}$ is the inclusion map, then

$$
\begin{aligned}
&< \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}([M] \otimes [N]), \Lambda(y) > \\
&\qquad =< [M].[N], \Lambda(y) > \\
&\qquad =< [M] \otimes [N], \triangle(\Lambda(y)) > \\
&\qquad =< [M] \otimes [N], (\Lambda \otimes \Lambda)(\triangle(y)) > \\
&\qquad = \sum < [M], \triangle(y') >< [N], \triangle(y'') > ,
\end{aligned}
$$

where

$$\triangle(y) = \sum y' \otimes y'' \ .$$

$$
\begin{aligned}
\Phi([M].[N])(y) &=< [M].[N], \Lambda(y) > \\
&= \sum \Phi([M])(y)\Phi([N])(y'') \\
&= (\Phi([M]) \otimes \Phi([N]))\left(\sum y' \otimes y''\right) \\
&= (\Phi([M]) \otimes \Phi([N]))\triangle(y) \\
&= (\Phi([M]).\Phi([N]))(y) \ .
\end{aligned}
$$

Hence Φ is a ring homomorphism.

Theorem 4.3. The map of Hopf rings $\Lambda : C \to R(S)$ is an isomorphism.

Proof. Consider the following

$$
\begin{array}{ccc}
C & & \\
\big\downarrow{\scriptstyle h^*} & & \\
C^* & \underset{\Phi}{\longleftarrow} & R(S)
\end{array} \quad .
$$

First we are going to show that the diagram commutes. Since $\Phi \circ \Lambda$ is a ring homomorphism, it is sufficient to show that

$$(\Phi \circ \Lambda)(y_k) = h^*(y_k) = \sum_{\pi \vdash k} y_\pi^* \ ,$$

for all $k \geq 1$. For any $y_\pi \in C$, where

$$\pi = \{\pi_1, \pi_2, \ldots, \pi_k\} \vdash k$$

we have

$$\begin{aligned}
&(\Phi \circ \Lambda(y_k))(y_\pi) = \Phi([1_{S_1}^{\otimes k} \otimes \eta_k])(y_\pi) \\
&=< [1_{S_1}^{\otimes k} \otimes \eta_k], \Lambda(y_\pi) > \\
&= \sum_{i=1}^{k} < [1_{S_1}^{\otimes i} \otimes \eta_k] \otimes [1_{S_1^{\otimes (k-i)}} \otimes \eta_{k-i}], \Lambda(y_\pi) > \\
&= \sum_{i=1}^{k} < \Lambda(y_i) \otimes \Lambda(y_{k-i}), \Lambda(y_\pi) > \\
&= \sum_{i=1}^{k} < (\Lambda \otimes \Lambda)(\triangle(y_k)), \Lambda(y_\pi) > \\
&=< \triangle(\Lambda(y_k)), \Lambda(y_\pi) > \\
&=< \Lambda(y_k), m(\Lambda(y_\pi)) > \\
&= \prod_{i=1}^{k} < \Lambda(y_k), \Lambda(y_{\pi_i}) > \\
&= \delta_{k\,\pi_k} = \sum_{k \vdash k} (y_\pi)^* = h^*(y_k) \ .
\end{aligned}$$

The third equality from the end is true because of the self–adjointness property in $R(S)$. Hence the diagram commutes. To finish the proof note that since h^* is an isomorphism, Φ is surjective. However, the rank of $R(S_k)$ is the same as the rank of C_{2k}^*. Hence Φ is an isomorphism. Thus $\Lambda = \Phi^{-1} \circ h^*$ is an isomorphism of Hopf rings. This completes the proof.

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$$\downarrow \qquad \downarrow$$

$$\downarrow \qquad \downarrow$$

$$\overline{\varphi} \qquad \overline{\Theta}$$

$$\Lambda$$