ON THE RIGHT-HAND DERIVATIVE OF INTEGRAL FUNCTIONS WITH A DISCONTINUOUS INTEGRAND

Joseph Bennish

California State University, Long Beach

Limits of the form

$$\lim_{x\to 0^+} \frac{1}{x}\int_0^x f(t)dt \tag{1}$$

arise in beginning calculus. This limit is simply the right-hand derivative at 0 of the integral function

$$\int_0^x f(t)dt.$$

Indeed, the Fundamental Theorem of Calculus, which connects differential and integral calculus, tells us that this limit is $f(0)$ for any integrable function f which is continuous at 0. But does this limit exist if f is discontinuous at 0? This is a natural question that can be investigated by students of calculus. We will find a class of discontinuous functions f for which the limit in (1) exists (Result 1), and another class of functions for which this limit does not exist (Result 2). Both of these results generalize examples which have appeared in the literature.

One example, namely $f(x) = \sin 1/x$ (or, what is essentially the same, $\cos 1/x$) has appeared in the literature (Problem E 1071, *American Mathematical Monthly,* 1954, p. 154; Problem E 1970, *op. cit.,* 1968, p. 678; Solution to problem 1112, *Mathematics Magazine* 55 (1982) 48) where it is shown that the limit in (1) is 0. All these solutions use the same argument (involving an integration by parts).

The function $f(x) = \sin(\ln x)$ was investigated by J. Klippert in [1], in which it was shown (again using an integration by parts) that the limit in question does not exist. He conjectured, based on these as well as other examples, that under certain conditions the limit in (1) must be zero. These conditions are that f is bounded and continuous on $(0, 1)$, and that "the zeros of f form a null sequence of distinct terms whose associated sequence of distances between consecutive zeros is asymptotically proportional to $1/n^\alpha$ for some $\alpha > 1$". In this paper we give a counter-example to this conjecture. However, we show, as a consequence of our first result, that this conjecture would be true provided the hypothesis were slightly strengthened.

In what follows we will assume that f is continuous on $(0, 1]$, and

$$\int_0^1 f(t)dt$$

exists as an improper integral. Also, as in Klippert's condition, the zeros of f form a null sequence converging to 0. We arrange the zeros a_n, $n = 1, 2, \ldots$, in decreasing order, and assume that the sign of f alternates on the intervals (a_{n+1}, a_n), $n \geq 1$. For definiteness, we take f positive (negative) on the interval (a_{n+1}, a_n) for n even (odd). Define

$$I_n = \left| \int_{a_{n+1}}^{a_n} f(t)dt \right|,$$

and let $d_n = a_n - a_{n+1}$. Also, we write $A_n \sim B_n$ if two sequences $\{A_n\}$ and $\{B_n\}$ are asymptotically proportional, i.e., A_n/B_n converges to a non-zero finite limit. Also, denote

$$\frac{\int_0^x f(t)dt}{x}$$

by $F(x)$.

<u>Counter-example to Klippert's Conjecture</u>. Let $a_n = 1/n$, and construct f so that I_n is equal to d_n if n is even and $d_n/2$ if n is odd. Then $d_n \sim n^{-2}$. The sequence d_n is decreasing, so

$$F(a_{2n}) = \frac{1}{a_{2n}} \sum_{k=n}^{\infty} (I_{2k} - I_{2k+1}) > \frac{1}{a_{2n}} \sum_{k=n}^{\infty} (d_{2k} - d_{2k}/2)$$

$$> \frac{1}{4a_{2n}} \sum_{k=n}^{\infty} (d_{2k} + d_{2k+1}) = 1/4,$$

since

$$a_n = \sum_{k=n}^{\infty} d_k.$$

Thus, the limit in question cannot be zero and, in fact, the limit can be shown to be 1/4 by showing that the average value of f on the interval (a_{n+2}, a_n) converges to 1/4 as $n \to \infty$. (Exercise: construct f satisfying Klippert's condition but with

$$\lim_{x \to 0^+} F(x)$$

non-existent.)

Result 1. We show now that the limit of $F(x)$ is zero as $x \to 0^+$ whenever f is bounded and $I_n/I_{n+1} \to 1$. Given $\epsilon > 0$, choose n so large that $|I_k/I_{k+1} - 1| < \epsilon$ for $k \geq n$. Choose x, $a_{n+1} \leq x \leq a_n$. Let M be a bound for $|f|$, and, for definiteness, assume that $n = 2m$ is even. If $F(x) \geq 0$ we have

$$F(x) \leq \frac{I_{2m} - I_{2m+1} + I_{2m+2} - \cdots}{a_{2m+1}} = \frac{1}{a_{2m+1}} \sum_{k=m}^{\infty} I_{2k+1} \left(\frac{I_{2k}}{I_{2k+1}} - 1 \right)$$

$$\leq \frac{\epsilon}{a_{2m+1}} \sum_{k=m}^{\infty} I_{2k+1} \leq \epsilon M \frac{\sum_{k=m}^{\infty} d_{2k+1}}{a_{2m+1}} \leq \epsilon M.$$

A similar argument shows that $F(x)$ is bounded below by $-\epsilon M$. Since $\epsilon > 0$ is arbitrarily small, the function $F(x)$ converges to 0 as $x \to 0^+$. In particular, this result shows that if $I_n \sim d_n$ in addition to Klippert's conditions, then the limit in (1) converges to 0. Indeed,

$$\frac{I_n}{I_{n+1}} = \frac{I_n/d_n}{I_{n+1}/d_{n+1}} \frac{d_n/n^{-\alpha}}{d_{n+1}/(n+1)^{-\alpha}} \frac{n^{\alpha}}{(n+1)^{\alpha}} \to 1, \quad \text{as} \quad n \to \infty.$$

The function $f(x) = \sin(1/x)$ satisfies these conditions. Although this approach is more involved than the solutions already cited, it makes the role of the cancellation clear.

In view of $a_n = 1/n\pi$, we have $d_n \sim n^{-2}$. By making the change of variables $u = 1/x$ in the integral I_n, a straightforward calculation gives $I_n/d_n \to 2/\pi$ as $n \to \infty$.

Result 2. We generalize the example $f(x) = \sin(\ln x)$ as follows: the limit in (1) does not exist if $I_n \sim d_n$ and $d_n/d_{n+1} \to L$, $L > 1$, as $n \to \infty$. (Note that the last condition is satisfied if $a_n \sim r^n$, $0 < r < 1$.) We will show that $F(a_n)$ is bounded below by a positive number if n is large and even. Let $n = 2m$. Then for n sufficiently large

$$F(a_{2m}) = \frac{1}{a_{2m}} \sum_{k=m}^{\infty} I_{2k+1} \left(\frac{I_{2k}/d_{2k}}{I_{2k+1}/d_{2k+1}} \frac{d_{2k}}{d_{2k+1}} - 1 \right) \geq \frac{c_1}{a_{2m}} \sum_{k=m}^{\infty} I_{2k+1}$$

$$\geq \frac{c_1}{a_{2m}} c_2 \sum_{k=m}^{\infty} d_{2k+1},$$

for some positive numbers c_1 and c_2. Moreover, for n large,

$$a_{2m} = \sum_{k=m}^{\infty} d_{2k} + \sum_{k=m}^{\infty} d_{2k+1} \leq (2L+1) \sum_{k=m}^{\infty} d_{2k+1},$$

so $F(a_{2m}) \geq c_1 c_2/(2L+1)$. Similarly, it can be shown that $F(a_n)$ is bounded above by a negative number if n is a large, odd number. Taken together these bounds show that the limit in (1) cannot exist.

For the function $f(x) = \sin(\ln x)$ we have $a_n = \exp(-\pi n)$, so $d_n/d_{n+1} = e^{\pi}$. Upon making the change of variables $u = \ln x$, a short calculation shows that $I_n/d_n = 1/2$.

The conditions of both results can be weakened, and this would be an instructive exercise for students who are taking a rigorous course in calculus. For instance, the hypothesis in the second result can be replaced by the following: $\liminf I_n/I_{n+1} > 1$, $\liminf I_n/d_n > 0$, and $\liminf d_{n+1}/d_n > 0$.

Reference

1. J. Klippert, "On the Right-Hand Derivative of a Certain Integral Function," *The American Mathematical Monthly,* 98 (1991), 751–752.