A NOTE ON A COUNTER EXAMPLE IN FINITE GROUPS

Kandasamy Muthuvel

University of Wisconsin-Oshkosh

It is known that if G is a finite abelian group and the number of elements of order 2 is not equal to 1, then the product of all elements of G is the identity (refer to p. 78, problem 43 of [2]). Garimella [1] gave a counter example to show that the conclusion of the above statement is not true for a non-abelian group of order 21. In this short note, we show that if G is a finite non-abelian group, then there is an arrangement $x_1, x_2, \ldots, x_n$ of all elements of G such that the product $x_1 x_2 \cdots x_n$ is a non-identity. The proof of this follows by finding elements a and b of G, say $a = x_1$ and $b = x_2$, such that $ab \neq ba$ and $abx_3 \cdots x_n$ or $bax_3 \cdots x_n$ is a non-identity. In fact if n is odd, we find such an arrangement by defining $x_1 = a$, $x_2 = b$, $x_3 = a^{-1}$, $x_4 = b^{-1}$, $x_{2i} = x_{2i-1}^{-1}$ for $3 \leq i \leq (n-1)/2$, and $x_n = e$.

On the other hand if G is a finite group such that

$$S = \{x \in G : x^2 = e\}$$

is a subgroup of G and $|S| \neq 2$, (in particular, if G is a group of odd order), there is an arrangement $x_1, x_2, \ldots, x_n$ of all elements of G such that the product $x_1 x_2 x_3 \ldots x_n$ is the identity (to prove this first, show that S is abelian, use p.78, problem 43 of [2] and the idea used in this note). The above result is not true for any finite group. For example, the product of all elements of S_3 (the set of all permutations of $\{1, 2, 3\}$) in any order is a non-identity.

References

1. R. Garimella, "A Counter Example in Group Theory," *Missouri Journal of Mathematical Sciences*, 3 (1991), 77–78.
2. I. N. Herstein, *Abstract Algebra*, 2nd ed., Macmillan, New York, 1990.