No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

41. [1992, 27] *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Define the sequence $\{L_n\}_{n=1}^{\infty}$ by $L_1 = a$, $L_2 = b$ and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$, where a and b are arbitrary integers. If $a = 1$ and $b = 2$, then $L_i = i$ for three consecutive integers i.

(i) Are there other values of a and b with this property?
(ii) Are there values of a and b such that $L_i = i$ for four consecutive values of i?
(iii)$*$ What happens if the 'consecutive' restriction is removed in (i) and (ii)?

Solution by N. J. Kuenzi and Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin (jointly).

(i) We will answer the more general question: Are there any other values of a and b such that three consecutive terms L_n, L_{n+1}, and L_{n+2} yield three consecutive integers i, $i+1$, and $i+2$?

Let $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$ denote the Fibonacci sequence. Then the sequence $L_1 = a$, $L_2 = b$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$ can be expressed in terms of the Fibonacci sequence by

$$L_{n+2} = aF_n + bF_{n+1} \quad \text{for} \quad n \geq 1.$$

If $L_n = i$, $L_{n+1} = i+1$, and $L_{n+2} = i+2$ then $2i+1 = i+2$ and $i = 1$. So $a = 1$ and $b = 2$ are the only values for a and b with the property that $L_i = i$ for three consecutive integers.

Next, suppose that

$$\begin{aligned} L_n &= aF_{n-2} + bF_{n-1} = 1 \\ L_{n+1} &= aF_{n-1} + bF_n = 2. \end{aligned}$$

Eliminating the variable b from these two equations yields

$$a(F_{n-2}F_n - F_{n-1}^2) = F_n - 2F_{n-1} = -F_{n-3}.$$

Since

$$F_{n-2}F_n - F_{n-1}^2 = (-1)^{n-1},$$

$$a = (-1)^n F_{n-3}.$$

Similarly,

$$b = (-1)^{n+1} F_{n-4}.$$

Example: If $a = F_7 = 13$ and $b = -F_6 = -8$, then $L_{10} = 1$, $L_{11} = 2$, and $L_{12} = 3$.

(ii) In part (i), we saw that if any three consecutive terms of the sequence yield three consecutive integers then the integers are 1, 2, and 3. Hence, it is not possible to have four consecutive integers as four consecutive terms of the sequence.

(iii) We will address the following question: Is it possible to have three distinct values $i < j < k$, other than 1, 2, 3 with $L_i = i$, $L_j = j$, and $L_k = k$?

Let i be the smallest subscript such that $L_i = i$ and let $c = L_{i+1} - L_i$.

(a) Suppose $c \geq 0$. Then

$$\begin{aligned}
L_{i+2} &= 2i + c \geq i + 1 \geq 2, \\
L_{i+3} &= 3i + 2c \geq i + 2, \\
L_{i+4} &= 5i + 3c \geq i + 4,
\end{aligned}$$

and

$$L_{i+n} > i + n \text{ for } n \geq 5.$$

So the only way we could have $L_j = j$ and $L_k = k$ would be to find values for i and c which would satisfy two of the four equations:

$$\begin{aligned}
i + c &= i + 1, \\
2i + c &= i + 2, \\
3i + 2c &= i + 3, \\
5i + 3c &= i + 4.
\end{aligned}$$

Exploring each of these possible cases yields only the solution $i = 1$, $c = 1$, $L_2 = 2$, and $L_3 = 3$.

So if $L_i = i$ and $c \geq 0$ it is not possible to have $L_i = i$, $L_j = j$, and $L_k = k$, for $i < j < k$, unless $i = 1$, $j = 2$, and $k = 3$.

(b) Suppose $c < 0$. Let j be the next smallest subscript such that $L_j = j$. Note that $j \geq i + 2 \geq 3$.

If the sequence of terms

$$L_i, L_{i+1}, \ldots, L_{i+n} = L_j$$

contained two consecutive negative terms, then all successive terms in the sequence would be negative and $L_j < 0$. So the sequence cannot contain two consecutive negative terms.

If the sequence of terms
$$L_i, L_{i+1}, \ldots, L_{i+n} = L_j$$
alternate in sign between positive and negative terms, then

$$i = L_i > L_{i+2} > L_{i+4} > \cdots > L_{i+n} = L_j = j.$$

But $i < j$ and so the sequence cannot just alternate in sign between positive and negative terms.

Since the sequence
$$L_i, L_{i+1}, \ldots, L_{i+n} = L_j$$
cannot alternate in sign and cannot contain two consecutive negative terms, it follows that $L_{j-1} \geq 0$.

If $L_{j-1} \neq 1$ then $L_{j+1} \neq j+1$,

$$\begin{aligned} &L_{j+1} \geq j, \\ &L_{j+2} \geq 2j > j+2, \end{aligned}$$

and
$$L_{j+n} > j + n \text{ for successive terms,}$$
(i.e., $L_k \neq k$ for all $k > j$).

This leaves only the case $L_i = i$, $L_{j-1} = 1$, $L_j = j$ to be considered. Working backward from L_j yields $L_j = j$, $L_{j-1} = 1$, $L_{j-2} = j-1$, $L_{j-3} = 2-j$, $L_{j-4} = 2j-3$, and in general

$$L_{j-n} = (-1)^n (jF_{n-1} - F_n).$$

But then
$$L_i = i = jF_{n-1} - F_n$$
for some even integer $n \geq 2$. So

$$\begin{aligned} i = jF_{n-1} - F_n &\geq (i+2)F_{n-1} - F_n \\ &\geq iF_{n-1} + 2F_{n-1} - F_n \\ &\geq i + 1. \end{aligned}$$

But this is a contradiction and so $L_{j-1} \neq 1$.

Hence it is not possible to have three distinct values $i < j < k$, other than 1, 2, 3, with $L_i = i$, $L_j = j$, and $L_k = k$.

(iv) In this section we address the following question:

Is it possible to have three terms of the sequence L_k, L_m, and L_n such that $L_k = i$, $L_m = i + 1$, and $L_n = i + 2$?

(a) Let $L_k = i$ and $L_{k+1} = 1$. Then $L_{k+2} = i + 1$ and $L_{k+3} = i + 2$. To find initial values for L_1 and L_2 simply work backward from $L_k = i$ and $L_{k+1} = 1$.

Example:

Suppose $L_6 = 18$ and $L_7 = 1$, then $L_8 = 19$ and $L_9 = 20$. Working backward yields $L_5 = -17$, $L_4 = 35$, $L_3 = -52$, $L_2 = 87$, and $L_1 = -139$.

(b) Let $L_n = i + 2$, and $L_{n+1} = -1$. Then $L_{n+2} = i + 1$ and $L_{n+3} = i$. To find initial values for L_1 and L_2, simply work backward from $L_n = i + 1$ and $L_{n+1} = -1$.

Example:

Suppose $L_6 = 18$ and $L_7 = -1$, then $L_8 = 17$ and $L_9 = 16$. Working backward yields $L_5 = -19$, $L_4 = 37$, $L_3 = -56$, $L_2 = 93$, and $L_1 = -149$.

(v) Finally we close with a question for the reader:

Is it possible to have three terms of the sequence L_k, L_m, and L_n with no two consecutive such that $L_k = i$, $L_m = i + 1$, and $L_n = i + 2$ for some i?

(i) and (ii) were also solved by the proposer.

42. [1992, 27] *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Find the general solution to the differential equation

$$\sum_{k=0}^{1990}(1990)^{k+1}y^{(k)} = 0,$$

where $y^{(k)}$ represents the kth derivative of y.

Solution by the proposer and Jayanthi Ganapathy, University of Wisconsin – Oshkosh, Oshkosh, Wisconsin (independently).

The characteristic equation for this differential equation is

$$(1990)^{1991}t^{1990} + (1990)^{1990}t^{1989} + \cdots + (1990)^2 t + 1990 = 0.$$

Next, dividing this equality by 1990 first, and then letting $u = 1990t$, we get

(1) $$u^{1990} + u^{1989} + \cdots + u + 1 = 0.$$

Now, using the fact that

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1),$$

the roots of (1) are 1990 distinct complex roots of unity

$$u_k = \cos\frac{2k\pi}{1991} + i\sin\frac{2k\pi}{1991}, k = 1, 2, \ldots, 1990.$$

Thus,

$$t_k = \frac{u_k}{1990}.$$

But, since t_{1991-k} is the complex conjugate of t_k, the general solution of the given differential equation is

$$y = \sum_{k=1}^{995} e^{\alpha_k x}\left(A_k\cos\beta_k x + B_k\sin\beta_k x\right),$$

where

$$\alpha_k = \frac{1}{1990}\cos\frac{2k\pi}{1991} \quad \text{and} \quad \beta_k = \frac{1}{1990}\sin\frac{2k\pi}{1991}.$$

Also solved by Russell Euler, Northwest Missouri State University, Maryville, Missouri.

43. [1992, 28] *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Let $n \geq 3$ be a positive integer and $m = \frac{n(n+1)}{2}$. Evaluate

$$\sum_{\substack{1 \leq a,b,c \leq n \\ a,b,c \text{ all distinct}}} \frac{abc}{m(m-a)(m-a-b)}.$$

Solution by Mangho Ahuja, Southeast Missouri State University, Cape Girardeau, Missouri, N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin, and Russell Euler, Northwest Missouri State University, Maryville, Missouri (independently).

$$\sum_{\substack{1 \leq a,b,c \leq n \\ a,b,c \text{ all distinct}}} \frac{abc}{m(m-a)(m-a-b)}$$

$$= \sum_{\substack{1 \leq a,b \leq n \\ a,b \text{ distinct}}} \frac{ab}{m(m-a)(m-a-b)} \sum_{\substack{1 \leq c \leq n \\ c \neq a,b}} c$$

$$= \sum_{\substack{1 \leq a,b \leq n \\ a,b \text{ distinct}}} \frac{ab}{m(m-a)(m-a-b)}(m-a-b)$$

$$= \sum_{\substack{1 \leq a,b \leq n \\ a,b \text{ distinct}}} \frac{ab}{m(m-a)}$$

$$= \sum_{1 \leq a \leq n} \frac{a}{m(m-a)} \sum_{\substack{1 \leq b \leq n \\ b \neq a}} b$$

$$= \sum_{1 \leq a \leq n} \frac{a}{m(m-a)}(m-a)$$

$$= \sum_{1 \leq a \leq n} \frac{a}{m} = \frac{1}{m} \sum_{1 \leq a \leq n} a = \frac{1}{m} m = 1.$$

Generalized Solution by Russell Euler, Northwest Missouri State University, Maryville, Missouri.

$$\sum_{\substack{1\le a_i\le n;\ i=1,\cdots,r\\ a_i\text{'s all distinct}}} \frac{a_1a_2\cdots a_r}{m(m-a_1)(m-a_1-a_2)\cdots(m-a_1-a_2-\cdots-a_{r-1})}$$

$$= \frac{1}{m}\sum_{a_1=1}^{n}\frac{a_1}{m-a_1}\sum_{\substack{a_2=1\\ a_2\ne a_1}}^{n}\frac{a_2}{m-a_1-a_2}\sum_{\substack{a_3=1\\ a_3\ne a_2,a_3\ne a_1}}^{n}\frac{a_3}{m-a_1-a_2-a_3}$$

$$\cdots\sum_{\substack{a_{r-1}=1\\ a_{r-1}\ne a_i;\ i=1,\cdots,r-2}}^{n}\frac{a_{r-1}}{m-a_1-a_2-\cdots-a_{r-1}}\sum_{\substack{a_r=1\\ a_r\ne a_i;\ i=1,\cdots,r-1}}^{n}a_r$$

$$= 1.$$

Also solved by the proposers.

44. [1992, 28] *Proposed by Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.*

It is well known that the set of all real numbers $\mathbb{R}$ is a field under ordinary addition and multiplication, the set of all positive real numbers $\mathbb{R}^+$ is a subgroup of the multiplicative group $\mathbb{R}^*$ (the set of all nonzero real numbers), $r^2 \in \mathbb{R}^+$ for all $r \in \mathbb{R}^*$, the characteristic of $\mathbb{R}$ is 0, and $\mathbb{R}^+ - \mathbb{R}^+ = \mathbb{R}$.

Prove the generalized result, "Let G be a subgroup of the multiplicative group F^* of a field F such that $f^2 \in G$ for all f in F^*. If the characteristic of F is not equal to 2, 3, and 5, then $G - G = F$."

Remark. The above result is not true if the characteristic of F is equal to 2, 3, or 5.

Solution by the proposer.

It is immediate that $G - G \subseteq F$. To show $F \subseteq G - G$, we first let $f \in F$ such that $f \neq \pm 1$. Then $1 + f$, $1 - f \in F^*$. Next, $2 \neq 0$ since the characteristic of F is not 2. Thus $4 = 2^2 \in G$. So $4^{-1} \in G$, since G is a group. Therefore,

$$\begin{aligned} f &= 4^{-1} \cdot 4f \\ &= 4^{-1}\left((1+f)^2 - (1-f)^2\right) \\ &= 4^{-1}(1+f)^2 - 4^{-1}(1-f)^2 \end{aligned}$$

so $f \in G - G$. Finally, we need to show $\pm 1 \in G - G$. First of all, $4^{-1} \neq \pm 1$ since the characteristic of F is not 3 and 5, respectively. Thus

$$1 = 4 \cdot 4^{-1} = \left(1 + 4^{-1}\right)^2 - \left(1 - 4^{-1}\right)^2,$$

so $1 \in G - G$. Similarly, $-1 \in G - G$. Thus $F \subseteq G - G$. Therefore, $G - G = F$ and this completes the proof.