SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

45. [1992, 88] *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Let α, β, and γ be the direction angles of a vector R. Without using Lagrange multipliers, show that

$$\cos\alpha + \cos\beta + \cos\gamma - 2\cos\alpha\cos\beta\cos\gamma \le \sqrt{2}.$$

Solution by the proposer.

If we let

$$\cos\alpha = \frac{a}{\sqrt{2}}, \quad \cos\beta = \frac{b}{\sqrt{2}}, \quad \cos\gamma = \frac{c}{\sqrt{2}},$$

then we need to show that

$$a + b + c - abc \le 2.$$

Thus, it is enough to prove that

$$4\bigl(4 - (a + b + c - abc)^2\bigr) \ge 0.$$

But, using

$$a^2 + b^2 + c^2 = 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 2,$$

we have

$$\begin{aligned} 4\bigl(4 - (a + b + c - abc)^2\bigr) &= (2 - 2ab)(2 - 2ac)(2 - 2bc) + 4(abc)^2 \\ &= \bigl(c^2 + (a - b)^2\bigr)\bigl(b^2 + (a - c)^2\bigr)\bigl(a^2 + (b - c)^2\bigr) + 4(abc)^2. \end{aligned}$$

This completes the proof. Note that equality holds if R is perpendicular to one of the axes and is making a 45-degree angle with each of the other two axes.

46. [1992, 88] *Proposed by Robert E. Kennedy and Curtis Cooper, Central Missouri State University, Warrensburg, Missouri.*

What is the leading digit of 2^{googol}?

Solution by Kevin Robinson, Messiah College, Grantham, Pennsylvania and the proposers.

The leading digit of 2^{googol} is 2. To see this, we observe that for any positive integer n,

$$2^n = 10^{\lfloor n \log 2 \rfloor} \cdot 10^{\{n \log 2\}},$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function, $\{\cdot\}$ denotes the fractional part operator, and $\log 2$ denotes the base 10 logarithm of 2. By the above identity, we have the leading digit of 2^n is the units digit of $10^{\{n \log 2\}}$. Using the computer algebra system DERIVE, the first 120 digits of $\log 2$ are

$$\begin{aligned} 0.&30102999566398119521\\ &37388947244930267681\\ &89881462108541310427\\ &46112710818927442450\\ &94869272521181861720\\ &40684477191430995379\,. \end{aligned}$$

Therefore,

$$10^{\{\text{googol} \log 2\}} = 10^{\{10^{100} \log 2\}} = 10^{.40684\ldots} = 2.\ldots$$

Thus, the leading digit of 2^{googol} is 2.

One incorrect solution was also received.

47. [1992, 88] *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Find all the solutions of

$$(x-1)x(x+1)(x+2)=-1.$$

Solution I by Bob Prielipp, University of Wisconsin - Oshkosh, Oshkosh, Wisconsin; Robert L. Doucette, McNeese State University, Lake Charles, Louisiana; Kevin Robinson, Messiah College, Grantham, Pennsylvania; Joseph E. Chance, University of Texas - Pan American, Edinburg, Texas; Frank Flanigan, San Jose State University, San Jose, California; Kandasamy Muthuvel, University of Wisconsin - Oshkosh, Oshkosh, Wisconsin; Hertag T. Freitag, Roanoke, Virginia; J. Sriskandarajah, University of Wisconsin Center, Richland Center, Wisconsin and the proposer.

We may construct the following collection of equivalent equalities.

$$(x-1)x(x+1)(x+2)=-1$$

$$x^4+2x^3-x^2-2x+1=0$$

$$(x^2+x-1)^2=0$$

$$x=\frac{-1+\sqrt{5}}{2}\ \text{(double root) or}\ x=\frac{-1-\sqrt{5}}{2}\ \text{(double root).}$$

Solution II by Mohammad K. Azarian, University of Evansville, Evansville, Indiana and J. Sriskandarajah, University of Wisconsin Center, Richland Center, Wisconsin.

From the substitution

$$x=\frac{u-1}{2},$$

we obtain

$$(x-1)x(x+1)(x+2)=\big(\frac{u-3}{2}\big)\big(\frac{u-1}{2}\big)\big(\frac{u+1}{2}\big)\big(\frac{u+3}{2}\big)=-1.$$

This implies that
$$u^4 - 10u^2 + 25 = 0.$$
Thus, $u^2 = 5$ is a double root. Hence, $u = \pm\sqrt{5}$. Therefore,
$$x = \frac{\pm\sqrt{5} - 1}{2},$$
where both of these roots are of multiplicity two.

Generalized Solution I by Joseph E. Chance and Donald P. Skow (jointly), University of Texas - Pan American, Edinburg, Texas.

Let k be an integer. Find all the solutions of
$$(x-k)(x-k+1)(x-k+2)(x-k+3) = -1.$$

Let
$$r = (x-k+1)(x-k+2).$$
Then
$$r = (x-k)^2 + 3(x-k) + 2.$$
Hence,
$$r - 2 = (x-k)^2 + 3(x-k) = (x-k)(x-k+3).$$
Thus, the original equation becomes
$$r(r-2) = -1.$$
This implies $r = 1$. Hence,
$$(x-k)^2 + 3(x-k) + 1 = 0$$
and so
$$x - k = \frac{-3 \pm \sqrt{5}}{2}.$$
Therefore,
$$x = \frac{-3 \pm \sqrt{5} + 2k}{2}.$$
If $k = 1$, we have the solution to problem 47.

Generalized Solution II by Frank Flanigan, San Jose State University, San Jose, California.

Let

$$P(x) = (x + a_1)(x + a_2)(x + b_1)(x + b_2) + k.$$

Suppose a_1, a_2, b_1, b_2, and k are elements of a field F. Assume

$$a_1 + a_2 = b_1 + b_2 = s.$$

Now, if $a = a_1 a_2$, $b = b_1 b_2$ and $y = x^2 + sx$, then

$$\begin{aligned}
P(x) &= (x^2 + sx + a)(x^2 + sx + b) + k \\
&= (y + a)(y + b) + k \\
&= y^2 + (a + b)y + (ab + k) \\
&= (y - \sigma)(y - \tau) \\
&= (x^2 + sx - \sigma)(x^2 + sx - \tau)
\end{aligned}$$

It is important that the roots σ, τ of the quadratic in y will lie in the original scalar field F of characteristic $\neq 2$ if and only if the discriminant

$$\triangle = \triangle_y = (a - b)^2 - 4k$$

is a square in F. This is decisive for the irreducibility of $P(x)$ in the ring $F[x]$ of polynomials.

Problem 47 can now be solved by taking $F = \mathbb{R}$ and $a_1 = 0$, $a_2 = 1$, $b_1 = -1$, $b_2 = 2$, and $k = 1$.

Also solved by N. J. Kuenzi, University of Wisconsin - Oshkosh, Oshkosh, Wisconsin; J. Sriskandarajah, University of Wisconsin Center, Richland Center, Wisconsin (second solution) and Donald P. Skow, University of Texas - Pan American, Edinburg, Texas. Frank Flanigan commented that his general solution to $P(x) = 0$ can be extended to the case when $P(x)$ involves the product of six, rather than four, suitable linear factors.

48. [1992, 89] *Proposed by Alvin Beltramo (student), Central Missouri State University, Warrensburg, Missouri.*

A standard deck of 52 cards is shuffled and then the cards are displayed, one at a time. Before each card is displayed, a person with a perfect memory guesses what each card is. How many cards can this person expect to guess correctly?

Solution I by Jenhua Tao, Central Missouri State University, Warrensburg, Missouri; Robert L. Doucette, McNeese State University, Lake Charles, Louisiana; K. L. D. Gunawardena, University of Wisconsin - Oshkosh, Oshkosh, Wisconsin; N. J. Kuenzi, University of Wisconsin - Oshkosh, Oshkosh, Wisconsin and Kevin Robinson, Messiah College, Grantham, Pennsylvania.

Define X_i to be the Bernoulli random variable for the ith card, i.e., $X_i = 0$ if the person guesses the ith card incorrectly and $X_i = 1$ if the person guesses the ith card correctly. Then

$$X = X_1 + X_2 + \cdots + X_{52}$$

is the number of cards this person guesses correctly. Thus,

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_{52}).$$

Since this person has perfect memory,

$$P(X_i = 1) = \frac{1}{52 - i + 1}$$

and

$$E(X_i) = \frac{1}{52 - i + 1}.$$

So,

$$E(X) = \frac{1}{52} + \frac{1}{51} + \frac{1}{50} + \cdots + 1 \doteq 4.538.$$

Generalized Solution I by K. L. D. Gunawardena, University of Wisconsin - Oshkosh, Oshkosh, Wisconsin.

Let $X_i = 1$ if the ith card is guessed correctly and $X_i = 0$ otherwise. Then

$$S = \sum_{i=1}^{52} X_i$$

denotes the total number of cards guessed correctly. Now suppose the person does not have a perfect memory, but he can remember the last k cards displayed prior to the displaying of the ith card. Thus,

$$P(X_i = 1) = \begin{cases} \frac{1}{53-i}, & 1 \leq i \leq k; \\ \frac{1}{52-k}, & k+1 \leq i \leq 52. \end{cases}$$

Therefore,

$$E(S) = \sum_{i=1}^{k} \frac{1}{53-i} + \sum_{i=k+1}^{52} \frac{1}{52-k} = 1 + \sum_{i=1}^{k} \frac{1}{53-i}.$$

Note that $k = 51$ corresponds to a person with a perfect memory.

Generalized Solution II by the proposer. Suppose there are n cards, numbered 1 to n in the deck. Then the expected number of correct guesses by a person with a perfect memory is

$$\sum_{k=1}^{n} \frac{1}{k}.$$

But,

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + \gamma + O\left(\frac{1}{n}\right),$$

where log is the natural logarithm and γ is Euler's constant. Thus, the expected number of correct guesses for a person with perfect memory, in a deck of n cards numbered 1 to n is asymptotic to $\log n$.

Also solved by the proposer. One incorrect solution was also received. Kevin Robinson and the proposer noted that the exact value is

$$\frac{14063600165435720745359}{3099044504245996706400}.$$