# AN EXAMPLE OF A FUNCTION IN LIP$_{\frac{1}{2}}([0, 2\pi])$

# WITH AN ABSOLUTELY NON-CONVERGENT FOURIER SERIES

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In a recent article in the **MJMS** [1], the construction, originally by Rudin & Shapiro [2], of a trigonometric series that converges uniformly to a function $f(t)$ which is continuous on $[0, 2\pi]$ was detailed. [1] includes a proof of the fact that the Fourier Series of $f(t)$ fails to converge absolutely.

In this paper we will take a closer look at this function $f(t)$. In fact, we will prove that $f \in \mathrm{Lip}_{\frac{1}{2}}([0, 2\pi])$, thus emphasizing the fact that $\frac{1}{2}$ is the best possible value of $\alpha$ in the established result of Bernstein's: "If $f \in \mathrm{Lip}_{\alpha}([0, 2\pi])$ for some $\alpha > \frac{1}{2}$, then the Fourier Series of $f$ is absolutely convergent" ([2], Chapter I).

We give below some definitions and results from [1] that are relevant to our discussion of $f(t)$ in this article.

1) We defined sequences of trigonometric polynomials $\{P_m(t)\}_0^\infty$, $\{Q_m(t)\}_0^\infty$ on $[0, 2\pi]$ inductively by letting $P_0(t) = 1 = Q_0(t)$. For $m \geq 0$,

$$P_{m+1}(t) = P_m(t) + e^{i2^m t} Q_m(t) \text{ and}$$
$$Q_{m+1}(t) = P_m(t) - e^{i2^m t} Q_m(t).$$

2) For $n \geq 1$, we defined the sequence of polynomials $\{T_n(t)\}_1^\infty$, by letting

$$T_n(t) = P_n(t) - P_{n-1}(t).$$

3) Finally, we defined a trigonometric series

$$\sum_{n=1}^{\infty} 2^{-n} T_n(t).$$

This series converges uniformly on $[0,2\pi]$ to a continuous function $f(t)$, which is the main topic of discussion in this article.

In [1], we made the following observations:

4) $|T_n(t)| \leq 2^{n/2}$.
5) $T_n(t)$ is a trigonometric polynomial of degree $2^n - 1$.

For the benefit of those readers who are unfamiliar with $\mathrm{Lip}_\alpha([0,2\pi])$, we provide some definitions at this point:

6) For $0 < \alpha \leq 1$, $\mathrm{Lip}_\alpha([0,2\pi])$ is the set of all $2\pi$ periodic, continuous complex valued functions $f(t)$ on $[0,2\pi]$ such that

$$\sup_{h\neq 0,t}\left\{\frac{|f(t+h)-f(t)|}{|h|^\alpha}\right\} < \infty.$$

([2], Chapter I).

Finally, before we proceed with our proof of $f \in \mathrm{Lip}_{\frac{1}{2}}([0,2\pi])$, we state a result of Bernstein's, that we will use in our discussion:

7) If $P(t)$ is a trigonometric polynomial of degree $n$ then

$$\sup_{t\in T}|P'(t)| \leq 2n\big(\sup_{t\in T}|P(t)|\big)$$

where $P'(t)$ is the derivative of $P$. ([2], Chapter I or [3], Chapter III, Sec. 13).

<u>Theorem</u>. $f \in \mathrm{Lip}_{\frac{1}{2}}([0,2\pi])$.

<u>Proof</u>. Fix $h \neq 0$. Choose an integer $N > 0$ such that $2^{-N} < |h| \leq 2^{1-N}$. Then

$$(A) \qquad |f(t+h)-f(t)| \leq |S_1(t)| + |S_2(t)|$$

where

$$S_1(t) = \sum_{n=1}^{N} 2^{-n}(T_n(t+h) - T_n(t))$$

and

$$S_2(t) = \sum_{n=N+1}^{\infty} 2^{-n}(T_n(t+h) - T_n(t)).$$

For any $n$ where $0 < \theta < h$,

$$\begin{aligned}
|T_n(t+h) - T_n(t)| &= |T_n'(t+\theta)||h|, \\
&\leq 2(2^n - 1)|h|\left(\sup_{t\in[0,2\pi]} |T_n(t)|\right) \\
&< 2(2^n)|h|2^{\frac{n}{2}},
\end{aligned}$$

using (7), (5), and (4). This implies

$$2^{-n}|T_n(t+h) - T_n(t)| < 2|h|2^{\frac{n}{2}}.$$

Hence,

$$\begin{aligned}
|S_1| &\leq 2|h|\left(\sum_{n=1}^{N} 2^{\frac{n}{2}}\right) \\
&= 2|h|2^{\frac{1}{2}}\frac{\left(2^{\frac{N}{2}} - 1\right)}{\sqrt{2} - 1} \\
&< 2|h|\sqrt{2}(\sqrt{2+1})\sqrt{2}\frac{1}{\sqrt{|h|}} \\
(*) \qquad &= M_1\sqrt{|h|}.
\end{aligned}$$

Here, $M_1$ is a constant. The last inequality follows by choosing $N$ so that $|h| \leq 2^{1-N}$ which implies that

$$2^{\frac{N}{2}} < \frac{\sqrt{2}}{\sqrt{|h|}}.$$

Also, using (4)

$$\begin{aligned}
|S_2(t)| &\leq \sum_{n=N+1}^{\infty} 2^{-n}(2)\big(2^{\frac{n}{2}}\big) \\
&= 2\left(\sum_{n=N+1}^{\infty} 2^{\frac{-n}{2}}\right) \\
&= \frac{2 \cdot 2^{-\left(\frac{N+1}{2}\right)}}{\left(1 - \frac{1}{\sqrt{2}}\right)} \\
&= 2\big(2^{-\frac{N}{2}}\big)(\sqrt{2}+1) \\
&< 2(\sqrt{2}+1)\sqrt{|h|} \\
(**) \qquad &= M_2\sqrt{|h|}
\end{aligned}$$

Here, $M_2$ is a constant. Also, the next to last inequality follows from the choice of $N$. Finally, from (A), $(*)$, and $(**)$, we get

$$|f(t+h) - f(t)| \leq (M_1 + M_2)\sqrt{|h|}.$$

Thus,

$$\frac{|f(t+h) - f(t)|}{\sqrt{|h|}} \leq M_1 + M_2.$$

Since $h \neq 0$ is arbitrary, we conclude that

$$\sup_{h\neq 0,t}\left\{\frac{|f(t+h)-f(t)|}{\sqrt{|h|}}\right\} < \infty.$$

This completes the proof of our theorem.

## *References*


1. J. Ganapathy, "Some Thoughts on the Absolute Convergence of a Trigonometric Series," *Missouri Journal of Mathematical Sciences*, 3 (1991), 2–11.

2. Y. Katznelson, *An Introduction to Harmonic Analysis*, John Wiley and Sons, 1968.

3. A. Zygmund, *Trigonometric Series*, Vol. I, Cambridge University Press, 1968.