SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

49. [1992, 145] *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Let

$$A = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{m} \sin\left(\frac{i\pi}{3}\right) \cos\left(\frac{j\pi}{3}\right) \csc\left(\frac{2^k \pi}{3}\right).$$

Show that

$$A \leq \frac{4\sqrt{3}}{3} \text{ if } m \text{ is odd and } A = 0 \text{ if } m \text{ is even.}$$

Solution I by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Our solution will use the following known results.

$$(1) \qquad \sum_{k=1}^{n} \cos kx = \sin \frac{nx}{2} \cos(n+1)\frac{x}{2} / \sin \frac{x}{2}$$

$$(2) \qquad \sum_{k=1}^{n} \sin kx = \sin \frac{nx}{2} \sin(n+1)\frac{x}{2} / \sin \frac{x}{2}$$

for every x that is not a multiple of 2π.
(For a proof of these results, see p. 366 of Apostol; *Mathematical Analysis: A Modern Approach to Advanced Calculus*; Addison-Wesley Publishing Company, Inc.; Reading Massachusetts; 1957.)

$$
\begin{aligned}
A &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m} \sin\left(\frac{i\pi}{3}\right)\cos\left(\frac{j\pi}{3}\right)\csc\left(\frac{2^k\pi}{3}\right) \\
&= \sum_{i=1}^{n} \sin\left(\frac{i\pi}{3}\right)\sum_{j=1}^{n}\cos\left(\frac{j\pi}{3}\right)\sum_{k=1}^{m}\csc\left(\frac{2^k\pi}{3}\right).
\end{aligned}
$$

If m is even,

$$
\sum_{k=1}^{m}\csc\left(\frac{2^k\pi}{3}\right) = 0
$$

so $A = 0$. If m is odd,

$$
\sum_{k=1}^{m}\csc\left(\frac{2^k\pi}{3}\right) = \csc\left(\frac{2\pi}{3}\right) = \frac{2\sqrt{3}}{3}.
$$

Thus,

$$
\begin{aligned}
A &= \frac{2\sqrt{3}}{3}\sum_{i=1}^{n}\sin\left(\frac{i\pi}{3}\right)\sum_{j=1}^{n}\cos\left(\frac{j\pi}{3}\right) \\
&= \frac{2\sqrt{3}}{3}\frac{\sin\frac{n\pi}{6}\sin\frac{(n+1)\pi}{6}}{\sin(\frac{\pi}{6})}\frac{\sin\frac{n\pi}{6}\cos\frac{(n+1)\pi}{6}}{\sin\left(\frac{\pi}{6}\right)} \\
&= \frac{4\sqrt{3}}{3}\sin\frac{n\pi}{6}\sin\frac{n\pi}{6}\sin\frac{(n+1)\pi}{3} \\
&\le \frac{4\sqrt{3}}{3}\cdot 1\cdot 1\cdot 1 = \frac{4\sqrt{3}}{3}.
\end{aligned}
$$

Solution II by the proposer.

It is known [1] that for any real number x

$$(1-\cos x)\left|\sum_{i=1}^{n}\cos ix\right|\left|\sum_{i=1}^{n}\sin ix\right| \leq 1.$$

Thus, for $x=\frac{\pi}{3}$ we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sin\left(\frac{i\pi}{3}\right)\cos\left(\frac{j\pi}{3}\right) \leq \left|\sum_{i=1}^{n}\sin\left(\frac{i\pi}{3}\right)\right|\left|\sum_{j=1}^{n}\cos\left(\frac{j\pi}{3}\right)\right| \leq 2. \tag{1}$$

Also, since

$$\cot(2^{k-1}x)-\cot(2^{k}x)=\csc(2^{k}x),$$

we have

$$\sum_{k=1}^{m}\csc(2^{k}x)=\cot x-\cot(2^{m}x),$$

and for $x=\frac{\pi}{3}$ we obtain

$$\sum_{k=1}^{m}\csc\left(\frac{2^{k}\pi}{3}\right)=\cot\frac{\pi}{3}-\cot\left(\frac{2^{m}\pi}{3}\right)=\begin{cases}0, & \text{if } m \text{ is even;}\\ \frac{2\sqrt{3}}{3}, & \text{if } m \text{ is odd.}\end{cases} \tag{2}$$

Finally, from (1) and (2) we achieve the desired result.

References

1. Problem # 379, *College Mathematics Journal*, M. K. Azarian (proposer) and Harry D'Souza (solver), 21 (1990), 248.

50. [1992, 145] *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

How many ways can a 3×1992 floor be tiled with 1×2 indistinguishable tiles?

Solution by Lamarr Widmer, Messiah College, Grantham, Pennsylvania and N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

We will let a_n denote the number of ways a $3 \times 2n$ floor can be tiled with dominoes (1×2 tiles). So our problem is to determine a_{996}.

We can easily check that $a_1 = 3$.

Now a tiling of a $3 \times 2n$ rectangle can be separated into two tilings of smaller $3 \times 2i$ and $3 \times 2j$ rectangles like this:

where $2i + 2j = 2n$, unless it has a row of horizontal dominoes through its center like this:

To see why this is true, note that if there is no such subdivision, then each of the vertical lines where this would be possible must have a domino lying across it. But since there are an even number of 1×1 squares on either side of such a vertical dividing line, it must actually have two dominoes lying across it. It is easy to see what goes wrong if we try to have these two dominoes in the top and bottom rows. Hence, one of them must be in the middle row. It is easy to check that there are exactly two possible tilings which have this horizontal row of dominoes through their center.

We first wish to deduce a general recursive formula for a_n. We will illustrate the general recursive principle by means of two examples. We see that there are 9 ways to tile a 3×4 floor by putting two tilings of the 3×2 floor side by side plus two ways to tile it with the row of horizontal dominoes through the center.

Hence $a_2 = 11$.

A similar line of reasoning can be used to compute a_5. The following diagram relates a_5 to its preceding terms.

Therefore,

$$a_5 = 3a_4 + 2a_3 + 2a_2 + 2a_1 + 2.$$

In general, we have that

$$a_n = 3a_{n-1} + 2a_{n-2} + 2a_{n-3} + 2a_{n-4} + 2a_{n-5} + \cdots + 2a_1 + 2.$$

From this relation we easily derive

$$a_n = 4a_{n-1} - a_{n-2}.$$

Using the method explained in most discrete mathematics texts, this second order linear recurrence relation yields the explicit formula

$$a_n = \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)(2+\sqrt{3})^n + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)(2-\sqrt{3})^n.$$

So we have

$$a_{996} = \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)(2+\sqrt{3})^{996} + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)(2-\sqrt{3})^{996} \doteq 3.603 \cdot 10^{569}.$$

Also solved by Stanley Rabinowitz, Westford, Massachusetts and the proposers. Rabinowitz and Kuenzi noted that

$$a_n = \left\lceil \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)(2+\sqrt{3})^n \right\rceil$$

for $n \geq 1$*. In addition, Rabinowitz counted and found that the number of ways to tile a* 3×1992 *floor with* 1×2 *indistinguishable tiles is*

3602927240267776227266301236128046740727140862109428584846
6187989288272765115658745939578174574051042816802245305844
6362912165834628327000163077343071409300022653365714290771
3843129066863598893786810870199167529198469052917243505662
2862052760951873433815826479063106067394965812084240925088
9591061988939195168722611435717987995627725683246896965503
4010414559073855237148010444891336978160899220084149304617
3795108177928277627669779323982081706670963802388825487599
4813205199045365601920341350977195999536039979037705807509
971867586054886929272135211907110350307848485081.

Rabinowitz also noted that the problem can be found in a couple of different places. One is a book by W. G. Kelley and A. C. Peterson entitled Difference Equations: An Introduction with Applications. This book is published by Academic Press, Inc. and the problem can be found on pages 89–91. The other place is Problem E2417 in the American Mathematical Monthly. The problem was proposed by Ioan Tomescu and a solution by D. Ž. Djoković was published in volume 81 (1974), pp. 522–523.

51. [1992, 145] *Proposed by Alvin Beltramo (student), Central Missouri State University, Warrensburg, Missouri.*

A standard deck of 52 cards is shuffled and two different denominations, e.g., king and five, are chosen. What is the probability that two cards, one from each denomination, are consecutive in the deck?

Solution by the proposer.

Label one of the denominations 1 and the other 2. Label the other cards in the deck 0. Now any arrangement of the deck will have a corresponding sequence of 44 0's, 4 1's, and 4 2's associated with it. Moreover, each sequence of 44 0's, 4 1's, and 4 2's will be the unique sequence associated with $44! \cdot 4! \cdot 4!$ different arrangements of the deck. We will proceed to count the number of different sequences of 44 0's, 4 1's, and 4 2's which do <u>not</u> have a 1 and 2 adjacent. If x denotes this number, then $x \cdot 44! \cdot 4! \cdot 4!$ arrangements of the deck do not have 2 cards, one from each denomination, side by side. Thus the probability that 2 cards, one from each denomination, are consecutive in the deck is

$$1 - \frac{x \cdot 44! \cdot 4! \cdot 4!}{52!}.$$

To compute x, the number of arrangements of 44 0's, 4 1's, and 4 2's which do not have a 1 and 2 adjacent, let us start by examing all

$$\binom{8}{4} = 70$$

arrangements of 4 1's and 4 2's. This list of 70 subdivides into 7 groups we will call groups 12, 121, 1212, 12121, 121212, 1212121, and 12121212. In the 12 group we have 2 sequences; they are

$$11112222 \quad \text{and} \quad 22221111.$$

In the 121 group we have 6 sequences; they are

11122221
11222211
12222111
22211112
22111122
21111222.

The 1212 group has 18 sequences

11122212	22211121
11122122	22211211
11121222	22212111
11222112	22111221
11221122	22112211
11211222	22122111
12221112	21112221
12211122	21122211
12111222	21222111.

Continuing in this manner the 12121 group has 18 elements, the 121212 group has 18 elements, the 1212121 group has 6 elements, and the 12121212 group has 2 elements. For each element in each group we must determine the number of ways we can distribute 44 0's so that the sequences of 44 0's, 4 1's, and 4 2's do not have 1 and 2 adjacent.

It turns out that if we determine the number of ways to distribute the 44 0's (so as to not have a 1 and 2 adjacent) in one representative from each of the seven groups, the other group's members have the same number of ways to distribute the 44 0's so that 1 and 2 are not adjacent.

Consider 11112222 from the 12 group. Placing 44 0's in this sequence so that it does not have a 1 and 2 adjacent would involve placing 43 0's in the 9 boxes, $\underbrace{\quad}$, depicted below.

$$\underbrace{\quad}1\underbrace{\quad}1\underbrace{\quad}1\underbrace{\quad}1\underbrace{\ 0\ }2\underbrace{\quad}2\underbrace{\quad}2\underbrace{\quad}2\underbrace{\quad}$$

Note that at least one 0 must go in the middle box. But by [1], the number of ways to put 43 0's in 9 boxes is

$$\binom{9-1+43}{9-1} = \binom{51}{8}.$$

By a similar discussion there are

$$\binom{51}{8}$$

ways to place 44 0's in 22221111 so that 1 and 2 are not adjacent. Thus, the number of sequences we get from the 12 group is

$$2 \cdot \binom{51}{8}.$$

Next take 11122221 from the 121 group. Placing 44 0's in this sequence so that it does not have a 1 and 2 adjacent would involve placing 42 0's in the 9 boxes depicted below.

$$\underbrace{\quad}1\underbrace{\quad}1\underbrace{\quad}1\underbrace{0}2\underbrace{\quad}2\underbrace{\quad}2\underbrace{\quad}2\underbrace{0}1\underbrace{\quad}$$

But the number of ways to do this is

$$\binom{50}{8}$$

so the number of sequences we get from the 121 group is

$$6 \cdot \binom{50}{8}.$$

Continuing in this manner we have the following chart.

Group	*Representative*	*# in Group*	*# per Repres.*
12	11112222	2	$\binom{51}{8}$
121	11122221	6	$\binom{50}{8}$
1212	11122212	18	$\binom{49}{8}$
12121	11222121	18	$\binom{48}{8}$
121212	11221212	18	$\binom{47}{8}$
1212121	12212121	6	$\binom{46}{8}$
12121212	12121212	2	$\binom{45}{8}$

Thus,

$$x = 2\binom{51}{8} + 6\binom{50}{8} + 18\binom{49}{8} + 18\binom{48}{8} + 18\binom{47}{8} + 6\binom{46}{8} + 2\binom{45}{8}.$$

Using the computer algebra system DERIVE,

$$x = 27061623270.$$

Putting this in

$$1 - \frac{x \cdot 44! \cdot 4! \cdot 4!}{52!}$$

and again simplifying using DERIVE, the required probability is

$$\frac{284622747}{585307450} \doteq 0.486279.$$

References

1. J. L. Mott, A. Kandel, and T. P. Baker, *Discrete Mathematics for Computer Scientists*, Reston Publishing Company, Reston, VA, (1983), 140–146.

52. [1992, 146] *Proposed by Dale Woods and Jin Chen, University of Central Oklahoma, Edmond, Oklahoma.*

(a) Find a closed form for the expression

$$\sum_{k=1}^{m} k\binom{2m}{m-k}.$$

(b)$*$ Let $n \geq 2$ be an integer. Find a closed form for the expression

$$\sum_{k=1}^{m} k^n\binom{2m}{m-k}.$$

Solution I to (a) by N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Let

$$a_m = \sum_{k=1}^{m} k\binom{2m}{m-k}.$$

We will show that

$$a_m = m\binom{2m-1}{m}$$

by showing that a_m satisfies the recurrence relation

$$a_{m+1} = 4a_m + \binom{2m}{m}$$

for $m \geq 1$ and that the solution to this recurrence relation with initial condition $a_1 = 1$ is

$$a_m = m\binom{2m-1}{m}.$$

To see the second claim first, let a_m be a sequence satisfying the recurrence relation

$$a_{m+1} = 4a_m + \binom{2m}{m}$$

for $m \geq 1$ with $a_1 = 1$. Then

$$a_1 = 1 = 1\binom{2-1}{1}.$$

Suppose that

$$a_n = n\binom{2n-1}{n}.$$

Then

$$\begin{aligned} a_{n+1} = 4a_n + \binom{2n}{n} &= 4n\binom{2n-1}{n} + \binom{2n}{n} \\ &= \frac{4n(2n-1)!}{n!(n-1)!} + \frac{(2n)!}{n!n!} \\ &= \frac{(2n)(2n)!}{n!n!} + \frac{(2n)!}{n!n!} \\ &= \frac{(2n+1)!}{n!n!} = (n+1)\binom{2n+1}{n+1}. \end{aligned}$$

So by the principle of mathematical induction,

$$a_m = m\binom{2m-1}{m}$$

for all positive integers m. Finally, to show that the sequence

$$a_m = \sum_{k=1}^{m} k\binom{2m}{m-k}$$

satisfies the recurrence relation, we will use the following identity

$$\binom{n+2}{j} = \binom{n}{j} + 2\binom{n}{j-1} + \binom{n}{j-2}.$$

Then

$$\begin{aligned}
a_{m+1} &= \sum_{k=1}^{m+1} k\binom{2m+2}{m+1-k} \\
&= \sum_{k=1}^{m-1} k\left[\binom{2m}{m+1-k} + 2\binom{2m}{m-k} + \binom{2m}{m-k-1}\right] + m(2m+2) + (m+1) \\
&= 2\sum_{k=1}^{m} k\binom{2m}{m-k} + \sum_{k=1}^{m-1} k\binom{2m}{m+1-k} + \sum_{k=1}^{m-1} k\binom{2m}{m-k-1} + 2m^2 + m + 1 \\
&= 2a_m + \binom{2m}{m} + \sum_{k=2}^{m-1} k\binom{2m}{m+1-k} + \sum_{k=1}^{m-1} k\binom{2m}{m-k-1} + 2m^2 + m + 1 \\
&= 2a_m + \binom{2m}{m} + \sum_{j=1}^{m-2} (j+1)\binom{2m}{m-j} + \sum_{i=2}^{m} (i-1)\binom{2m}{m-i} + 2m^2 + m + 1 \\
&= 2a_m + \binom{2m}{m} + \sum_{j=1}^{m-2} j\binom{2m}{m-j} + \sum_{i=2}^{m} i\binom{2m}{m-i} + \binom{2m}{m-1} + 2m^2 - m \\
&= 2a_m + \binom{2m}{m} + \sum_{j=1}^{m} j\binom{2m}{m-j} + \sum_{i=1}^{m} i\binom{2m}{m-i} \\
&= 4a_m + \binom{2m}{m}.
\end{aligned}$$

Solution II to (a) by the proposers.

To prove the (a) part we need the fact that for $1 \le k \le m$,

$$(m-k)\binom{2m}{m-k} = 2m \cdot \binom{2m-1}{m-1-k}. \tag{1}$$

This can be shown by using the factorial form of the binomial coefficients. Next, writing (1) in a slightly different form, we have that

$$k\binom{2m}{m-k} = m\binom{2m}{m-k} - 2m\binom{2m-1}{m-1-k}. \tag{2}$$

Now summing both sides of (2) as k ranges between 1 and m and using the assumption that

$$\binom{2m-1}{-1} = 0,$$

we have that

$$\begin{aligned}
\sum_{k=1}^{m} k\binom{2m}{m-k} &= \sum_{k=1}^{m}\left(m\binom{2m}{m-k} - 2m\binom{2m-1}{m-1-k}\right) \\
&= m\sum_{k=1}^{m}\binom{2m}{m-k} - 2m\sum_{k=1}^{m}\binom{2m-1}{m-1-k} \\
&= m\sum_{k=0}^{m-1}\binom{2m}{k} - 2m\sum_{k=0}^{m-2}\binom{2m-1}{k} \\
&= m\frac{2^{2m}-\binom{2m}{m}}{2} - 2m\left(2^{2m-2} - \binom{2m-1}{m-1}\right) \\
&= m\cdot 2^{2m-1} - \frac{m}{2}\binom{2m}{m} - m\cdot 2^{2m-1} + m\cdot 2\binom{2m-1}{m-1} \\
&= \frac{m}{2}\binom{2m}{m}.
\end{aligned}$$

Solution III to (a) by the proposers. We need to use the summation by parts formula

$$\sum_{k=1}^{m} b_k(a_{k+1}-a_k) + \sum_{k=1}^{m} a_k(b_k - b_{k-1}) = a_{m+1}b_m - a_1 b_0.$$

(This identity can be proved by expanding the summation.) Let $b_k = k$ and

$$a_k = \sum_{j=1}^{k-1} \binom{2m}{m-j}.$$

(Note that $a_1 = 0$.) Then using summation by parts

$$\sum_{k=1}^{m} k\binom{2m}{m-k} + \sum_{k=1}^{m}\sum_{j=1}^{k-1} \binom{2m}{m-j} = m\sum_{j=1}^{m} \binom{2m}{m-j}.$$

Thus

$$\sum_{k=1}^{m} k\binom{2m}{m-k} + \sum_{k=1}^{m-1} k\binom{2m}{k} = m\sum_{k=0}^{m-1} \binom{2m}{k}.$$

But

$$k\binom{2m}{k} = 2m\binom{2m-1}{k-1}$$

so we have

$$\sum_{k=1}^{m} k\binom{2m}{m-k} + 2m\sum_{k=1}^{m-1} \binom{2m-1}{k-1} = m\sum_{k=0}^{m-1} \binom{2m}{k}.$$

Therefore,

$$\begin{aligned}
\sum_{k-1}^{m} k\binom{2m}{m-k} &= m\sum_{k=0}^{m-1}\binom{2m}{k} - 2m\sum_{k=1}^{m-1}\binom{2m-1}{k-1} \\
&= m\sum_{k=0}^{m-1}\binom{2m}{k} - 2m\sum_{k=0}^{m-2}\binom{2m-1}{k} \\
&= m\cdot\frac{2^{2m}-\binom{2m}{m}}{2} - 2m\left(\frac{2^{2m-1}}{2} - \binom{2m-1}{m-1}\right) \\
&= m\cdot 2^{2m-1} - \frac{m}{2}\binom{2m}{m} - m\cdot 2^{2m-1} + 2m\binom{2m-1}{m-1} \\
&= -\frac{m}{2}\binom{2m}{m} + m\binom{2m}{m} \\
&= \frac{m}{2}\binom{2m}{m}.
\end{aligned}$$

Part (b) of Problem 52 still remains open.