A TRIGONOMETRIC DERIVATION OF POINT-TO-LINE AND POINT-TO-PLANE DISTANCE FORMULAS

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The aim of the note is to obtain the distance formulas in an elementary fashion with a minimal amount of calculations. Why should there be interest in this very old problem which had been resolved in so many different ways? Admittedly, one can use partial derivatives to minimize the square of the distance between the given point and a variable point on the line or plane, but the method is far beyond the means of precalculus. Furthermore, algebraic attempts to minimize this function involve a large number of tedious computations. Some students suggest a straightforward approach that includes the following steps: write an equation of the line which is perpendicular to the given line and passes through the given point; solve the system of equations and thus, find the point of intersection of the lines; to complete the solution, use the distance formula between two points. This is easier said than done, and soon we realize that the volume of calculations is overwhelming. Another procedure offered in some texts reduces the problem to finding the distance between parallel lines. One has to admit that this method is not encouraging either.

On the other hand, our trigonometric method is elementary and short, and it works for both the line and plane. From the given point $P(x_0, y_0)$ draw a perpendicular PQ to the x-axis and a perpendicular PR to the given line $(L) : Ax + By + C = 0$. Assuming (L) is not vertical, let $H(x_0, y_1)$ be the point of intersection of PQ and (L). We notice that the points P and H have the same first coordinate x_0 because PQ is perpendicular to the x-axis. Also, we observe that the coordinates of H satisfy the equation of the line (L), that is,

$$Ax_0 + By_1 + C = 0 \quad \text{or} \quad By_1 = -Ax_0 - C. \tag{1}$$

For the slope of (L) we can write the formulas

$$m = -\frac{A}{B} \quad \text{and} \quad m = \tan\theta,$$

where θ is the angle between the line and the x-axis. Therefore, $\tan\theta = -A/B$. Since $PQ \perp Ox$ and $PR \perp (L)$, then $\angle RPH = \pi - \theta$. Denote by d and c the lengths of PR and PH, respectively. From $\triangle RPH$ we have

$$d = c|\cos(\pi - \theta)| = |y_0 - y_1| \cdot |\cos\theta|.$$

From the trigonometric identity $\sec^2\theta = \tan^2\theta + 1$ we get

$$cos^2\theta = \frac{1}{\tan^2\theta + 1} \quad \text{and} \quad |\cos\theta| = \frac{1}{\sqrt{\tan^2\theta + 1}}.$$

Hence,

$$\begin{aligned} d = \frac{|y_0 - y_1|}{\sqrt{\tan^2\theta + 1}} &= \frac{|y_0 - y_1|}{\sqrt{\frac{A^2}{B^2} + 1}} \\ &= \frac{|By_0 - By_1|}{\sqrt{A^2 + B^2}}, \end{aligned}$$

and it remains to substitute here By_1 with its expression (1), to obtain the well known formula for the distance from a point to a line:

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

The simplicity of the trigonometric method is just one of its advantages. Its generality enables us to find, in the same manner, the distance formula from a point to a plane. To calculate the distance from the point $P(x_0, y_0, z_0)$ to the plane $Ax + By + Cz + D = 0$, we draw from P a perpendicular PQ to the coordinate plane $x0y$ and a perpendicular PR to the given plane. Assuming the given plane is not parallel to the z-axis, the point H where PQ crosses this plane has coordinates (x_0, y_0, z_1) that satisfy the equation

$$Ax_0 + By_0 + Cz_1 + D = 0,$$

whence

$$(2) \qquad Cz_1 = -Ax_0 - By_0 - D.$$

The length of PH equals $|z_0 - z_1|$, and from the right triangle RPH we find

$$d = |z_0 - z_1| \cdot |\cos\theta|,$$

where d is the length of PR and $\theta = \angle RPH$. Note that θ is the angle between PQ (which is parallel to the z-axis) and PR (which is perpendicular to the given plane.) Hence,

$$\cos\theta = \frac{c}{\sqrt{A^2 + B^2 + C^2}},$$

since A, B, and C are the components of a vector normal to the plane. By virtue of (2),

$$d = \frac{|Cz_0 - Cz_1|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$