ITERATIONS ON CONVEX QUADRILATERALS

Tamela Underwood

Southern Illinois University at Carbondale

Mangho Ahuja

Southeast Missouri State University

1. Introduction. The object of this paper is to study the effect of the repeated applications of a particular process $\mathcal{P}$, when it is performed on an arbitrary (convex) quadrilateral. The process is described below.

Process $\mathcal{P}$. Given a quadrilateral $ABCD$, we construct squares on the sides AB, BC, CD, and DA [Fig. 1]. All four squares are constructed on the outside of $ABCD$. Let P_1, Q_1, R_1, and S_1 denote the centers of the squares on the sides AB, BC, CD, and DA, respectively. By joining the centers of the squares a new quadrilateral $P_1Q_1R_1S_1$ is obtained. The process of obtaining quadrilateral $P_1Q_1R_1S_1$ from quadrilateral $ABCD$ is defined as the process $\mathcal{P}$.

We will denote $P_1Q_1R_1S_1$ by $\mathcal{P}[ABCD]$ and also by Π_1. In general $P_nQ_nR_nS_n$ and Π_n will denote the quadrilateral obtained by applying the process n times. In Proposition 1 we will prove that the quadrilateral $P_1Q_1R_1S_1$ has the following properties:

(i) $P_1R_1 = Q_1S_1$, i.e. the diagonals are equal, and

(ii) P_1R_1 is perpendicular to Q_1S_1, i.e. the diagonals are perpendicular.

We note that properties (i) and (ii) are not sufficient to make $P_1Q_1R_1S_1$ a square. For our purpose we may define a square as follows. A quadrilateral $PQRS$ is a square if it has the following three properties:

(i) $PR = QS$,

(ii) PR is perpendicular to QS,

(iii) the diagonals PR and QS bisect each other.

We have seen that just one application of process $\mathcal{P}$ transforms an arbitrary quadrilateral into one which satisfies two of the three properties for a square. One wonders what effect repeated applications of $\mathcal{P}$ would have on $ABCD$. Since Π_1 satisfies (i) and (ii), it is obvious that every quadrilateral Π_n will also satisfy (i) and (ii). Let M_n and N_n denote the midpoints of the diagonals P_nR_n and Q_nS_n respectively, and let s_n denote the distance M_nN_n. Since every Π_n satisfies (i) and (ii), Π_n will be a square if and only if the diagonals bisect each other, i.e. if and only if $s_n = 0$.

We also note that with each application of process $\mathcal{P}$ the quadrilateral increases in size. Let d_n denote the length of the diagonals (both diagonals have the same length) of Π_n, and let $s'_n = \frac{s_n}{d_n}$. To measure the effect of $\mathcal{P}$ on Π_n, we should compare quadrilaterals of the same size. This means that we should not compare s_n with s_{n+1}, but we should compare s'_n with s'_{n+1}. We would like to know if the quadrilaterals Π_n obtained by n applications of $\mathcal{P}$ would 'approach a square' i.e. whether s'_n goes to 0 as n tends to infinity. We will explore this question and also study other effects of $\mathcal{P}$.

2. We now give an analytic proof of Proposition 1. For a proof using rotations of triangles, see [1].

Proposition 1. Let $ABCD$ be an arbitrary polygon and $P_1Q_1R_1S_1$ be the polygon obtained by applying process $\mathcal{P}$. Then the diagonals P_1R_1 and Q_1S_1 are equal and perpendicular.

Proof. Choosing the axes as shown in Fig. 2, let the coordinates of the vertices be given by $A = (0,0)$, $B = (a,b)$, $C = (c,d)$, and $D = (e,0)$. Without loss of generality assume $b > d$. Also $c > a$ (convexity). We construct square $BCB'C'$ on the outside of BC and denote its center by Q_1. Let CN and $C'N'$ be perpendiculars on the vertical line through $B(a,b)$. The right triangles BCN and $C'BN'$ are congruent ($BC = BC'$, also angles CBN and $C'BN'$ are complementary, making angles BCN and $C'BN'$ equal). Then $CN = BN' = c - a$, and $BN = C'N' = b - d$. This gives $C' = (a + b - d, -a + b + c)$, and the midpoint of CC',

$$Q_1 = \left(\frac{a+b+c-d}{2}, \frac{-a+b+c+d}{2}\right).$$

Using the same method we find the centers of the other squares,

$$P_1 = \left(\frac{a-b}{2}, \frac{a+b}{2}\right),$$

$$R_1 = \left(\frac{c+d+e}{2}, \frac{-c+d+e}{2}\right)$$

$$S_1 = \left(\frac{e}{2}, \frac{-e}{2}\right).$$

The lengths and the slopes of the diagonals are as follows:

$$P_1R_1 = \frac{1}{2}\sqrt{(a-b-c-d-e)^2 + (a+b+c-d-e)^2} = Q_1S_1.$$

The slope of P_1R_1 is

$$\frac{a+b+c-d-e}{a-b-c-d-e}$$

and the slope of Q_1S_1 is

$$\frac{-a+b+c+d+e}{a+b+c-d-e}.$$

This shows that the diagonals P_1R_1 and Q_1S_1 are equal and perpendicular, which completes the proof.

From Proposition 1, we note that M_1, N_1, the midpoints of P_1R_1 and Q_1S_1 are given by

$$M_1 = \frac{1}{4}(a-b+c+d+e, a+b-c+d+e)$$

and

$$N_1 = \frac{1}{4}(a+b+c-d+e, -a+b+c+d-e),$$

and the distance $s_1 = M_1N_1$ is given by

$$s_1^2 = \left(\frac{-b+d}{2}\right)^2 + \left(\frac{a-c+e}{2}\right)^2.$$

If the original quadrilateral $ABCD$ was a parallelogram, then $b = d$, and $c - e = a$. This would make $s_1 = 0$, and $P_1Q_1R_1S_1$ would be a square.

We now set the stage to study the process $\mathcal{P}$ and the properties of the quadrilateral Π_n. Since the diagonals of Π_1 are perpendicular, they will be chosen as the coordinate axes [Fig. 3]. As mentioned earlier, $P_2Q_2R_2S_2$ will denote the quadrilateral obtained by applying $\mathcal{P}$ to $P_1Q_1R_1S_1$. Let M_2, N_2 denote the midpoints of the diagonals P_2R_2 and Q_2S_2 respectively, and d_n denote the length (both diagonals have the same length) of the diagonals of the quadrilateral Π_n.

Proposition 2.

(i) The diagonals of Π_2 make a 45° angle with the diagonals of Π_1, and both sets of diagonals intersect at the same point,

(ii) The line segments M_1N_1, M_2N_2 are equal and bisect each other,

(iii) $d_2 = \sqrt{2}d_1$.

Proof. Let the coordinates of P_1, Q_1, R_1, and S_1 be given by $P_1 = (0, y_1)$, $Q_1 = (x_1, 0)$, $R_1 = (0, y_1 - d_1)$, $S_1 = (x_1 - d_1, 0)$ (see Fig. 3). Then $M_1 = (0, y_1 - \frac{1}{2}d_1)$, $N_1 = (x_1 - \frac{1}{2}d_1, 0)$. Using the same method as in Proposition 1, the vertices of Π_2 are obtained as:

$$P_2 = \left(\frac{x_1+y_1}{2}, \frac{x_1+y_1}{2}\right), \quad Q_2 = \left(\frac{x_1-y_1+d_1}{2}, \frac{-x_1+y_1-d_1}{2}\right),$$

$$R_2 = \left(\frac{x_1+y_1}{2} - d_1, \frac{x_1+y_1}{2} - d_1\right), \quad S_2 = \left(\frac{x_1-y_1-d_1}{2}, \frac{-x_1+y_1+d_1}{2}\right).$$

Then

$$M_2 = \left(\frac{x_1+y_1-d_1}{2}, \frac{x_1+y_1-d_1}{2}\right), \quad N_2 = \left(\frac{x_1-y_1}{2}, \frac{-x_1+y_1}{2}\right).$$

We note that the points P_2 and R_2 satisfy the equation $y = x$, whereas the points Q_2 and S_2 satisfy the equation $y = -x$. In other words, the diagonals of Π_2 lie along the lines $y = \pm x$. Thus they intersect at $(0, 0)$ and make a $45°$ angle with the previous set of diagonals. Using distance formula,

$$M_1N_1^2 = \left(x_1 - \frac{1}{2}d_1\right)^2 + \left(y_1 - \frac{1}{2}d_1\right)^2,$$

and

$$M_2N_2^2 = \left(y_1 - \frac{1}{2}d_1\right)^2 + \left(x_1 - \frac{1}{2}d_1\right)^2.$$

Thus, $M_1N_1 = M_2N_2$. Also, M_1N_1 and M_2N_2 have the same midpoint

$$G = \left(\frac{2x_1 - d_1}{4}, \frac{2y_1 - d_1}{4}\right).$$

Lastly,

$$\begin{aligned}
d_2^2 &= P_2R_2^2 \\
&= \frac{1}{4}\left(\left(x_1 + y_1 - (x_1 + y_1 - 2d_1)\right)^2 + \left((x_1 + y_1) - (x_1 + y_1 - 2d_1)\right)^2\right) \\
&= \frac{1}{4}\left(4d_1^2 + 4d_1^2\right) = 2d_1^2.
\end{aligned}$$

Hence, $d_2 = \sqrt{2}d_1$. This completes the proof.

3. We come back to the question of whether the quadrilaterals Π_n 'approach a square.' Here is our main theorem.

<u>Theorem</u>. With repeated applications of the process $\mathcal{P}$ the quadrilaterals Π_n 'approach a square,' i.e. s'_n approaches 0, as n goes to infinity.

<u>Proof</u>. From Proposition 2 we have $M_1N_1 = M_2N_2$, i.e. $s_1 = s_2$. This means that $s_n = s_1$ for all n. Also, $d_2 = \sqrt{2}d_1$ implies $d_n = (\sqrt{2})^{n-1}d_1$. Hence,

$$s'_n = \frac{s_n}{d_n} = \frac{s_1}{(\sqrt{2})^{n-1}d_1}.$$

The quantity s'_n goes to 0 as n tends to infinity. In other words, the quadrilaterals Π_n approach a square with repeated applications of the process $\mathcal{P}$.

4. We now turn our attention to other effects of the process $\mathcal{P}$. The following observations are based on the information provided by Propositions 1 and 2.

1. The diagonals of every quadrilateral Π_n intersect at the same point.
2. The diagonals of every quadrilateral Π_n make a $45°$ angle with the diagonals of the previous one.
3. The centroid of every quadrilateral Π_n is the same point. We know that if the vertices of a quadrilateral are given by (x_i, y_i) for $i = 1$ to 4, then its centroid G is given by,

$$G = \left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}\right).$$

   The centroid of a quadrilateral is also the midpoint of the line segment joining the midpoints of the diagonals, i.e. G is the midpoint of M_1N_1. But from Proposition 2, we learn that M_1N_1 and M_2N_2 have the same midpoint, hence, the quadrilaterals Π_1 and Π_2 have the same centroid. It follows that every quadrilateral Π_n has the same centroid.

5. To summarize: With each application of the process $\mathcal{P}$, the diagonals rotate by $45°$, and their lengths decrease by a factor of $\sqrt{2}$. But the diagonals of every quadrilateral intersect at the same point. Moreover while the quadrilaterals increase in size, their growth in each direction is such that the centroid remains the same. Thus, each application of process $\mathcal{P}$ leaves two points fixed, the point where the diagonals intersect and the centroid G. Lastly, the quadrilateral Π_n never becomes a square, only its "non-squareness" as measured by s'_n tends to 0.

References

1. I. M. Yaglom, *Geometric Transformations*, Random House Inc., New York, NY, 1962.
2. T. Underwood, Masters Degree Thesis, Southeast Missouri State University, Cape Girardeau, MO, May 1992.
3. J. R. Smart, *Modern Geometries*, Brooks Cole Publishing Co., Monterey, CA, 1973.

Figure 1.

Figure 2.

Figure 3.