SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

47. [1992, 88; 1993, 94] *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Find all the solutions of

$$(x-1)x(x+1)(x+2) = -1.$$

Comment by the editor.

Due to an error by the editor, a remark by the proposer was omitted from the solution to the problem. At the time the problem was proposed, the proposer noted that if x is an integer, the factorization provided in the solution indicates that the product of four consecutive integers is one less than the square of an integer.

57. [1993, 90] *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Let s and k be positive integers. Evaluate

$$\lim_{n\to\infty}\prod_{i=1}^{k}\sum_{j=1}^{n} j^i\left(\frac{s}{n}\right)^{i+1}.$$

Solution I by Joseph E. Chance, University of Texas–Pan American, Edinburg, Texas; Seung–Jin Bang, Albany, California; and the proposer.

$$\sum_{j=1}^{n} j^i\left(\frac{s}{n}\right)^{i+1} = \sum_{j=1}^{n}\left(\frac{js}{n}\right)^{i}\frac{s}{n}$$

is an upper Riemann sum for

$$\int_0^s x^i\,dx.$$

Since the product of sums is finite and each sum has a limit,

$$\begin{aligned}
\lim_{n\to\infty}\prod_{i=1}^{k}\sum_{j=1}^{k} j^i\left(\frac{s}{n}\right)^{i+1} &= \prod_{i=1}^{k}\lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{js}{n}\right)^{i}\frac{s}{n} \\
&= \prod_{i=1}^{k}\int_0^s x^i\,dx \\
&= \prod_{i=1}^{k}\frac{s^{i+1}}{i+1} \\
&= \frac{s^{k(k+3)/2}}{(k+1)!}.
\end{aligned}$$

Solution II by N. J. Kuenzi, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin and Frank J. Flanigan, San Jose State University, San Jose, California.

It is known that

$$\sum_{j=1}^{n} j^i$$

can be represented as a polynomial in n of degree $i+1$ with leading coefficient $1/(i+1)$.

(See for example: R. S. Luthar, "A Simple Way of Evaluating $\sum_{i=1}^{k} i^n$", *Pi Mu Epsilon Journal*, 6 (1976), 282–284.) Using this fact we can represent $\sum_{j=1}^{n} j^i$ in the form

$$\sum_{j=1}^{n} j^i = \left(\frac{1}{i+1}\right)n^{i+1} + P(n),$$

where $P(n)$ is a polynomial of degree i. Since

$$\frac{P(n)}{n^{i+1}} \to 0 \quad \text{as} \quad n\to\infty,$$

it follows that

$$\sum_{j=1}^{n} \frac{j^i}{n^{i+1}} \to \frac{1}{i+1} \quad \text{as} \quad n \to \infty.$$

Hence,

$$\lim_{n\to\infty} \prod_{i=1}^{k} \sum_{j=1}^{n} j^i \left(\frac{s}{n}\right)^{i+1} = \prod_{i=1}^{k} s^{i+1} \left(\lim_{n\to\infty} \sum_{j=1}^{n} \frac{j^i}{n^{i+1}}\right)$$

$$= \prod_{i=1}^{k} \frac{s^{i+1}}{i+1} = \frac{s^{k(k+3)/2}}{(k+1)!}.$$

Also solved by Jayanthi Ganapathy, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin.

58. [1993, 90] *Proposed by Kandasamy Muthuvel, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin.*

Let G be a proper subgroup of $\mathbb{R}$, the reals under addition. Prove that $\mathbb{R}$ and the complement of G have the same cardinality.

Solution I by the proposer.

Let $|G^c|$ denote the cardinality of the complement of G and H denote the set of all finite linear combinations of elements from G^c with integer coefficients (that is, an element of H is of the form

$$\sum_{i=1}^{m} n_i x_i,$$

where m is a positive integer, the n_i's are integers and the x_i's are elements from G^c). Then $G^c \subseteq H$ and H is a subgroup of $\mathbb{R}$. Suppose $|G^c| < |\mathbb{R}|$. Then G is not a subset of H, because if $G \subseteq H$, then $\mathbb{R} = G \cup G^c \subseteq H$ and consequently $|H| = |\mathbb{R}|$, which contradicts the fact that $|G^c| < |\mathbb{R}|$. Since G is not a subset of H and H is not a subset of G, there exist elements $g \in G$ and $h \in H$ such that $g \notin H$ and $h \notin G$. If $g + h \in G$, then $h = g + h - g \in G$, but $h \notin G$. Hence $g + h \notin G$. Similarly $g + h \notin H$. Hence $g + h \notin G \cup H$. But $g + h \in \mathbb{R}$. This contradicts the fact that $\mathbb{R} = G \cup H$. Thus $|G^c| = |\mathbb{R}|$.

Solution II by Frank J. Flanigan, San Jose State University, San Jose, California.

We will deduce the above assertion after establishing the following theorem.

Theorem. Let G be a subset of an infinite set S whose complement $G^c = S - G$ satisfies $|G^c| \geq |G|$. Then $|G^c| = |S|$.

Proof. The theorem follows from

$$|S| = |G \cup G^c| = |G| + |G^c| = |G^c|,$$

since $|G^c|$ is necessarily infinite. (The second equality follows since the sets G and G^c are disjoint.)

Now if G is a proper subgroup of $\mathbb{R}$, then the standard Lagrange decomposition of $\mathbb{R}$ as a union of cosets $x + G$ shows that the complement G^c contains at least one such coset $x + G$. Thus

$$|G^c| \geq |x + G| = |G|,$$

so the theorem applies.

Comment. This approach does not involve uncountability, intermediate cardinalities, the continuum hypothesis, Hamel bases, etc. The assertion holds for any proper subgroup of any infinite group.

59. [1993, 91] *Proposed by Ollie Nanyes, Bradley University, Peoria, Illinois.*

Find a topology τ_1 for the real line $\mathbb{R}^1$ such that:

1) $(\mathbb{R}^1, \tau_1)$ is a second countable, metrizable space and
2) there is a homeomorphism

$$f\colon (\mathbb{R}^1, \tau_1) \to (\mathbb{R}^2, \tau_2),$$

where τ_2 is the product topology $\tau_1 \times \tau_1$.

Solution by the proposer.

Let τ_1 be the topology generated by $[q, x)$ where q and x are rational. Notice that these basis elements are closed and open. Clearly $(\mathbb{R}^1, \tau_1)$, denoted by $\mathbb{R}_q$, is Hausdorff, regular and second countable. Therefore, $\mathbb{R}_q$ is metrizable by the Urysohn metrization theorem.

However, we can find a basis $\mathcal{B}$ for $\mathbb{R}_q$,

$$\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i,$$

where:

1) for all i, $\mathcal{B}_i$ is a disjoint, countable cover of $\mathbb{R}^1$ by non-empty closed-open sets,

2) for all i, if $B \in \mathcal{B}_i$, there is a countable, disjoint cover $\mathcal{B}(B)$ of B such that

$$B = \bigcup_{B' \in \mathcal{B}(B)} B'$$

and

$$\mathcal{B}_{i+1} = \bigcup_{B \in \mathcal{B}_i} \{\mathcal{B}(B)\}$$

and

3) for every collection $\{B_i\}$ where, for all i, $B_i \in \mathcal{B}_i$ and $B_{i+1} \subset B_i$, then $\cap_i B_i$ is a single point set. Then, $\mathbb{R}_q$ is homeomorphic to $\mathbb{Z}^{\mathbb{Z}}$ in the product (Tychonoff) topology by the map $g\colon \mathbb{R}_q \to \mathbb{Z}^{\mathbb{Z}}$ by

$$g(x) = (n_1, n_2, \ldots, n_k, \ldots)$$

where

$$x \in \bigcap_{j=n_k,\ k \in \mathbb{Z}^+} B_i^j$$

(with $B_i^k \in \mathcal{B}_i$ for all i).

So, let τ_2 be the product topology for $\mathbb{R}_q \times \mathbb{R}_q$; since $\mathbb{Z}^{\mathbb{Z}}$ is homeomorphic to $\mathbb{Z}^{\mathbb{Z}} \times \mathbb{Z}^{\mathbb{Z}}$, $\mathbb{R}_q$ is homeomorphic to $\mathbb{R}_q \times \mathbb{R}_q$.

We can show that the necessary basis $\mathcal{B}$ exists by induction: let $\mathcal{C} = \{[a,b) \mid a, b \text{ are rational }\}$ and enumerate the rationals by q_i. Our induction hypothesis is that $\mathcal{B}_n \in \mathcal{C}$. If we let:

$$\mathcal{B}_1 = \bigcup_i [i, i+1),$$

then our hypothesis is satisfied for $i = 1$. Now assume that $\mathcal{B}_n \in \mathcal{C}$. We will describe the construction of $\mathcal{B}_{n+1}$. If $B \in \mathcal{B}_n$, then $B = [a,b)$ by hypothesis. Let $\mathcal{B}'(B) = \{[c,d) \in C \mid a \leq c < d < b \text{ with } |d - c| < 1/n\}$. Certainly if $q_n \in [a,b)$ there is a rational d_n such that $[q_n, d_n) \in \mathcal{B}'(B)$. Choose a countably infinite family of disjoint members of $\mathcal{B}'(B)$ whose union covers B such that $[q_n, d_n) \in \mathcal{B}(B)$ if $q_n \in B$. Call this collection $\mathcal{B}(B)$. Define

$$\mathcal{B}_{n+1} = \bigcup_{B \in \mathcal{B}_n} \{\mathcal{B}(B)\}.$$

It is now easy to check that

$$\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$$

is our basis which meets requirements 1, 2, and 3. (It turns out that $\mathbb{R}_q$ is homeomorphic to the irrationals in the normal Euclidean subspace topology since the irrationals are homeomorphic to $\mathbb{Z}^{\mathbb{Z}}$. (See F. Willard, *General Topology,* Addison-Wesley, 1970, Reading, Massachusetts, exercise 24K.)

Acknowledgement by the proposer. I would like to thank an anonymous referee and John Duncan who explained the solution to me.

One incorrect solution was also received.

60. [1993, 91] *Proposed by Alvin Tinsley, Central Missouri State University, Warrensburg, Missouri.*

Suppose a unit square has its left–hand corner at the origin and its sides along the x– & y–axes. Initially, place the base of an equilateral triangle with unit sides on the x–axis between 0 and 1. Slide the triangle to the left and up, always keeping the two vertices of the base in contact with the x– & y–axes until the base of the equilateral triangle is on the y–axis between 0 and 1. What is the equation of the locus of points determined by the third vertex of the triangle?

Solution I by N. J. Kuenzi, University of Wisconsin–Oshkosh, Oshkosh, Wisconsin; Leon Hall, University of Missouri–Rolla, Rolla, Missouri; and Rhonda McKee, Central Missouri State University, Warrensburg, Missouri.

Let t be the y–coordinate of the vertex A on the y–axis. Then $\sqrt{1-t^2}$ is the x–coordinate of the vertex B on the x–axis. For $0 < t < 1$, the slope of the line through A and B is $-t/\sqrt{1-t^2}$, the slope of the perpendicular bisector of side AB is $\sqrt{1-t^2}/t$, and the coordinates of the midpoint M of side AB are $(\sqrt{1-t^2}/2, t/2)$. (See figure 1.)
Now the third vertex C lies on the perpendicular bisector of side AB at a distance of $\sqrt{3}/2$ units from M. So if (x, y) are the coordinates of vertex C we have

$$(1) \qquad y - \frac{t}{2} = \frac{\sqrt{1-t^2}}{t}\left(x - \frac{\sqrt{1-t^2}}{2}\right)$$

and

$$(2) \qquad (y - \frac{t}{2})^2 + \left(x - \frac{\sqrt{1-t^2}}{2}\right)^2 = \frac{3}{4}.$$

Substituting the right hand side of (1) for $(y - t/2)$ in (2) and simplifying yields

$$\left(x - \frac{\sqrt{1-t^2}}{2}\right)^2 = \frac{3t^2}{4}. \tag{3}$$

Suppose the initial position for vertex C was quadrant I. Then from equations (3) and (1) we get the following parametric equations:

$$x = \frac{\sqrt{3}}{2}t + \frac{\sqrt{1-t^2}}{2}$$

and

$$y = \frac{t}{2} + \frac{\sqrt{3}}{2}\sqrt{1-t^2}, \quad 0 \le t \le 1.$$

(See figure 2.)
Using these equations we can eliminate the radical and solve for t,

$$t = \sqrt{3}x - y.$$

Substituting $\sqrt{3}x - y$ for t in the parametric equation for y and simplifying yields,

$$\sqrt{3}y - x = \sqrt{1 - (\sqrt{3}x - y)^2}.$$

Squaring both sides and simplifying yields

$$4x^2 - 4\sqrt{3}xy + 4y^2 = 1$$

where $1/2 \le x \le 1$ and $1/2 \le y \le 1$.
Next suppose the initial position for vertex C was quadrant IV. Then from equations (3) and (1) we get the following parametric equations:

$$x = -\frac{\sqrt{3}}{2}t + \frac{\sqrt{1-t^2}}{2}$$

and

$$y = \frac{t}{2} - \frac{\sqrt{3}}{2}\sqrt{1-t^2}, \quad 0 \leq t \leq 1.$$

(See figure 3.)
Using the same technique as in the previous case we get

$$\sqrt{3}x + y = -t$$

and

$$4x^2 + 4\sqrt{3}xy + 4y^2 = 1$$

where $-\sqrt{3}/2 \leq x \leq 1/2$, $-\sqrt{3}/2 \leq y \leq 1/2$, and ($x < 0$ or $y < 0$).

Solution II by the proposer; Joseph E. Chance, University of Texas–Pan American, Edinburg, Texas; and J. Sriskandarajah, University of Wisconsin Center–Richland, Richland Center, Wisconsin.

Consider the figure

The triangle slides as θ moves from 0 to 1. The equations of the non–base sides of the triangle are

$$y = \tan\left(\frac{2\pi}{3} - \frac{\pi}{2}\theta\right)(x - \cos\frac{\pi}{2}\theta)$$

and

$$y - \sin\frac{\pi}{2}\theta = \tan\left(\frac{\pi}{3} - \frac{\pi}{2}\theta\right)x.$$

Solving both equations simulataneously (and using some trig identities), the locus of the third vertex of the triangle is

$$(x, y) = \left(\cos(\frac{\pi}{3} - \frac{\pi}{2}\theta), \sin(\frac{\pi}{3} + \frac{\pi}{2}\theta)\right).$$

Finding θ in terms of x, we have

$$\frac{\pi}{2}\theta = \frac{\pi}{3} - \arccos x.$$

Thus

$$y = \sin\left(\frac{2\pi}{3} - \arccos x\right)$$

$$= \frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{1-x^2}.$$

Hence

$$2y - \sqrt{3}x = \sqrt{1-x^2}$$

so

$$4x^2 - 4\sqrt{3}xy + 4y^2 - 1 = 0.$$