SOME CRITERIA FOR THE HOMOTOPY METHOD FOR THE TRIDIAGONAL EIGENPROBLEM

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Abstract. In this paper, we shall give some criteria which guarantee the safety of choosing a diagonal starting matrix of the homotopy method for the symmetric tridiagonal eigenproblem.

1. Introduction. The homotopy continuation method can be used to solve the symmetric eigenvalue problem:

$$Ax = \lambda x \tag{1}$$

where A is an $n \times n$ real symmetric tridiagonal matrix of the form

$$A = \begin{pmatrix} \alpha_1 & \beta_2 & & & \\ \beta_2 & \alpha_2 & \beta_3 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{n-1} & \alpha_{n-1} & \beta_n \\ & & & \beta_n & \alpha_n \end{pmatrix}. \tag{2}$$

In (2), if some $\beta_i = 0$, then $\mathbb{R}^n$ is clearly decomposed into two complementary subspaces invariant under A. Thus, the eigenproblem is decomposed in an obvious way into two smaller subproblems. Therefore we will assume that each $\beta_i \neq 0$. That is, A is *unreduced.*

Consider the homotopy, $H: \mathbb{R}^n \times \mathbb{R} \times [0,1] \to \mathbb{R}^n \times \mathbb{R}$, defined by

$$\begin{aligned} H(x,\lambda,t) &= (1-t)\begin{pmatrix} \lambda x - Dx \\ \frac{x^T x - 1}{2} \end{pmatrix} + t \begin{pmatrix} \lambda x - Ax \\ \frac{x^T x - 1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \lambda x - [(1-t)D + tA]x \\ \frac{x^T x - 1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \lambda x - A(t)x \\ \frac{x^T x - 1}{2} \end{pmatrix} \end{aligned} \tag{3}$$

where $A(t) = (1-t)D + tA$. Here, D is the initial matrix, a specially chosen symmetric tridiagonal matrix such that $A(t)$ is unreduced for t in $(0,1]$. It can be seen that the solution set of $H(x,\lambda,t) = 0$ in (3) consists of disjoint smooth curves $(x(t),\lambda(t))$, each joins an eigenpair of D to one of A. We call each of these curves a *homotopy curve* or an *eigenpath* and its component $\lambda(t)$ an *eigenvalue path.* Thus, by following the eigenpaths emanating from the eigenpairs of D at $t=0$, we can reach all the eigenpairs of A at $t=1$.

The theoretical aspect of the continuation approach to the eigenvalue problems has been studied in [1,2,3,9]. A first attempt was made in [7] to make the method computationally efficient. In [7], the initial matrix D was chosen as $D = \mathrm{diag}\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$. An efficient parallel and robust algorithm was given in [6]. Evidenced by the numerical results, the algorithm is strongly competitive with other methods in terms of speed, accuracy and orthogonality, and leads in speed in almost all cases. In [6],

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \tag{4}$$

where

$$D_1 = \begin{pmatrix} \alpha_1 & \beta_2 & & & \\ \beta_2 & \alpha_2 & \beta_3 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{k-1} & \alpha_{k-1} & \beta_k \\ & & & \beta_k & \alpha_k \end{pmatrix}, \quad D_2 = \begin{pmatrix} \alpha_{k+1} & \beta_{k+2} & & & \\ \beta_{k+2} & \alpha_{k+2} & \beta_{k+3} & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{n-1} & \alpha_{n-1} & \beta_n \\ & & & \beta_n & \alpha_n \end{pmatrix}.$$

It is desirable to choose D as a diagonal matrix, consisting of the diagonal part of A, rather than the form in [6]. If D is a diagonal matrix, then eigenvalues and corresponding eigenvectors of D are immediately available. Thus, the work of solving the eigenproblem of D is saved. In [7], Li and Rhee showed that this strategy worked very well for certain matrices, such as $[1,i,1]$, $i = 1,2,\ldots,n$. That is, if we choose $D = \mathrm{diag}\{1,2,\ldots,n\}$ in solving the eigenproblem of the matrix $[1,i,1]$, the eigenpaths are still very flat and easy to follow. However, this strategy breaks down when we consider the eigenproblem of tridiagonal matrices $[1,2,1]$. The eigenpaths are rather difficult to follow. In this paper, we shall give some criteria which guarantee the safety of choosing a diagonal starting matrix D.

2. Criteria. Let $(A)^1$ denote the $(n-1)\times(n-1)$ matrix obtained by deleting the first row and column of A, and $(A)_1$ the $(n-1)\times(n-1)$ matrix obtained by deleting the last row and column of A. Let $\lambda_i(A)$ denote the ith smallest eigenvalue of A.

<u>Theorem 1</u>. If $\alpha_i < \alpha_{i+1}$, $i = 1,2,\ldots,n-1$ and if there exists a constant c, $0 < c \le 1$ such that

$$(A)^1 - (A)_1 - c \min_{1\le i\le n-1}(\alpha_{i+1}-\alpha_i) I$$

is positive semidefinite then

$$\min_{1\le i\le n-1}(\lambda_{i+1}-\lambda_i)\ge c\min_{1\le i\le n-1}(\alpha_{i+1}-\alpha_i)$$

where $\lambda_i=\lambda_i(A)$.

Proof. Since A is symmetric, so are $(A)_1$ and $(A)^1$. Let

$$\mu_1\le\mu_2\le\cdots\le\mu_{n-1}$$

and

$$\delta_1\le\delta_2\le\cdots\le\delta_{n-1}$$

be the eigenvalues of $(A)^1$ and $(A)_1$, respectively, then by Cauchy's interlacing theorem [10],

$$\lambda_1\le\mu_1\le\lambda_2\le\cdots\le\mu_{n-1}\le\lambda_n \tag{5}$$

$$\lambda_1\le\delta_1\le\lambda_2\le\cdots\le\delta_{n-1}\le\lambda_n. \tag{6}$$

Since $(A)^1=(A)_1+c\alpha I+[(A)^1-(A)_1-c\alpha I]$, and $(A)^1-(A)_1-c\alpha I$ is positive semidefinite, where

$$\alpha=\min_{1\le i\le n-1}(\alpha_{i+1}-\alpha_i),$$

by the Courant–Fisher maximum characterization [12],

$$\lambda_i((A)^1)\ge\lambda_i((A)_1+c\alpha I)\quad\text{for any } i,\ 1\le i\le n-1$$

i.e.,

$$\mu_i-\delta_i\ge c\alpha>0,\quad 1\le i\le n-1.$$

By (5) and (6),

$$\lambda_1\le\delta_1\le\mu_1\le\lambda_2\le\cdots\le\delta_{n-1}\le\mu_{n-1}\le\lambda_n.$$

Hence,

$$\lambda_{i+1}-\lambda_i\ge\mu_i-\delta_i\ge c\alpha,\quad 1\le i\le n-1$$

and

$$\min_{1\le i\le n-1}(\lambda_{i+1}-\lambda_i)\ge c\min_{1\le i\le n-1}(\alpha_{i+1}-\alpha_i).$$

Corollary 1. If

$$(A)_1 - \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_{n-1} \end{pmatrix} = (A)^1 - \begin{pmatrix} \alpha_2 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix},$$

then

$$\min_{1\le i\le n-1}(\lambda_{i+1}-\lambda_i) \ge \min_{1\le i\le n-1}(\alpha_{i+1}-\alpha_i). \tag{7}$$

Proof. (7) follows immediately from Theorem 1, since

$$(A)^1 - (A)_1 - \begin{pmatrix} \alpha_2-\alpha_1 & & \\ & & \ddots \\ \alpha_n-\alpha_{n-1} & & \end{pmatrix} = 0.$$

Let $A(t) = (1-t)D + tA$, where D is a diagonal matrix consisting of the diagonal elements of A. Then we have the following corollary.

Corollary 2.

$$\min_{1\le i\le n-1}(\lambda_{i+1}(t)-\lambda_i(t)) \ge c \min_{1\le i\le n-1}(\alpha_{i+1}-\alpha_i), \quad t\in[0,1].$$

Proof.

$$\begin{aligned} &(A(t))^1 - (A(t))_1 - \alpha I = t((A)^1-(A)_1-\alpha I) \\ &+ (1-t)\mathrm{diag}(\alpha_2-\alpha_1-\alpha, \alpha_3-\alpha_2-\alpha, \cdots, \alpha_n-\alpha_{n-1}-\alpha), \end{aligned} \tag{8}$$

where

$$\alpha = c\min_{1\le i\le n-1}(\alpha_{i+1}-\alpha_i), \quad 0<c\le 1.$$

Clearly, the second term of the right hand side of (8) is positive semidefinite and the first term is positive semidefinite by assumption. Hence, $(A(t))^1-(A(t))_1-\alpha I$ is positive semidefinite for $t\in[0,1]$. By Theorem 1,

$$\min_{1\le i\le n-1}(\lambda_{i+1}(t)-\lambda_i(t)) \ge c\min_{1\le i\le n-1}(\alpha_{i+1}-\alpha_i) \quad t\in[0,1].$$

Remark. If $(\beta_{i+1} - \beta_i)^2 < (\alpha_i - \alpha_{i-1})(\alpha_{i+1} - \alpha_i)$, $i = 2, 3, \ldots, n-1$ and $\alpha_i < \alpha_{i+1}$, $i = 1, 2, \ldots, n-1$, then A satisfies the conditions in Theorem 1.

If A satisfies the conditions in Theorem 1, we may choose the initial matrix D as a diagonal matrix consisting of the diagonal elements of A, then $A(t)$ is an unreduced symmetric tridiagonal matrix and the eigenvalue curves are not only distinct, but also very well separated. There is a lower bound between any two eigenvalue curves so that the eigenvalue curves are easy to follow.

Example 1. $A = [1, i, 1]$, where $i = 1, 2, \ldots, 20$. If we let $D = \mathrm{diag}\{1, 2, \ldots, 20\}$, then all the eigenvalue curves are very well separated. See Figure 1.

Figure 1. The eigenvalue curves of $[1, i, 1]$ matrix with $D = [0, i, 0]$.

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