APPLICATIONS OF COLORABILITY TO LINKS,

WILD KNOTS AND NON–COMPACT KNOTS

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Abstract. In this paper, mod n colorability is discussed and applied to tame links, tangles, wild knots and non–compact knots. In particular, colorability is used to obtain elementary non–unknotting criteria for wild knots and non–untangling criteria for tangles.

1. Introduction. Colorability has long been an elementary tool for determining whether a knot diagram represents a non–trivial knot. If a knot has a diagram which can be colored mod n, then the knot cannot be ambiently deformed into the standard "unknotted" circle. This is a result that is often presented to "honors" level undergraduate students. Colorability mod n has been discussed in a paper by the author [1], Kauffman [2] and Fox [3]. In order to keep this paper self–contained we will discuss coloring and give applications to links (a disjoint collection of knots), non–compact knots (embeddings of the real line in 3–space) and wild knots.

2. Definitions. A *knot* is a simple closed curve K in $\mathbb{R}^3$. A *tame knot* will be a polyhedral simple closed curve in $\mathbb{R}^3$. A *link* will be a finite disjoint union of tame knots. A map $f\colon X \to Y$ is *proper* if for every compact $C \subseteq Y$, $f^{-1}(C)$ is compact in X. A *non–compact knot* is the image of a proper polyhedral embedding of $\mathbb{R}^1$ in $\mathbb{R}^3$. The intuition is that $\pm\infty$ "goes to" ∞ in $\mathbb{R}^3$. Two knots (possibly wild or non–compact) K and K' are *equivalent* if there is an orientation–preserving homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K) = K'$. A (non–compact) knot K is *unknotted* if K is equivalent to the simple closed curve in $\mathbb{R}^3$ described by: $\{(x, y, 0) \mid x^2 + y^2 = 1\}$ ($\{(x, 0, 0) \mid x \in \mathbb{R}\}$ if K is non–compact). Otherwise, K is said to be *knotted*.

Let K be a simple closed curve in $\mathbb{R}^3$. If K is not equivalent to a p. l. knot, we say that K is *wild*. Let $p \in K$. p is a *wild point* of K for every polyhedral ball B containing p, $(B, B \cap K)$ is not a standard ball pair; that is, $B \cap K$ is not a properly embedded unknotted arc in B. Convention: when we say "*knot*", we will mean "*tame knot*" unless otherwise specified. Figures 1, 2, and 3 show examples of a knot, non–compact knot and a wild knot, respectively.

A *diagram* of a knot or a link (tame, non–compact, or wild with at most a finite number of wild points) will be the image of a projection of the knot or link onto a plane in $\mathbb{R}^3$ together with crossing information at the singular points (points with multiple preimages); the crossing information tells us which strand "goes over" which. A *regular projection* is a projection in which there are only a finite number (in the case where K is non–compact, the singular set is, at most, a countable discrete set) of singular points, every singular point is the result of a non–tangential intersection and every singular point is a double point. Figures 1, 2, and 3 are examples of regular projections. Figure 4 shows examples of what is not allowed in a regular projection.

Two diagrams D_1 and D_2 of a knot or link K (K can be non–compact or have a finite number of wild points) are said to be equivalent if D_1 can be deformed into D_2 by deformations of the plane (which respects crossing information of D_1) and by a finite number of moves (and their inverses), which are illustrated in Figure 5. These moves are called the *Reidemeister moves*. Note that in each move, the diagram remains fixed outside of the given circle. The following is well known:

Theorem 2.1. Two knots or links are equivalent if and only if all of their diagrams are equivalent.

Proof. See reference [4]. These ideas were introduced in 1932 by Reidemeister; see reference [5].

Research Problem. Find some sort of a Reidemeister theorem for non–compact knots or for knots with one wild point. One difficulty of this problem is illustrated by Figures 6 and 7; the non–compact knot in Figure 6 is known to be knotted (proved in reference [6]; we will give another proof) and the non–compact knot in Figure 7 is unknotted. Both of these non–compact knots have diagrams which can be deformed into the "trivial" non–compact knot by a countable number of Reidemeister moves.

3. Colorability mod n of Knot or Link Diagrams. A knot (possibly non–compact) diagram is said to be *colorable* mod n if

1) at each overcrossing we can assign integers (known as *colors*) a, b, c where b is assigned to the overcrossing, a is assigned to one of the other strands and $c = 2b - a \pmod{n}$ is assigned to the remaining strand, (see Figure 8) and

2) at least two distinct integers $\pmod{n}$ are used.

A knot K is said to be *colorable* mod n if K has a diagram which is colorable mod n.

Figure 9 shows that the trefoil knot is colorable mod 3.

Theorem 3.1. Colorability mod n is a knot and link diagram invariant.

Sketch of Proof. One just checks that colorability mod n is preserved by the Reidemeister moves and their inverses.

It follows from Theorem 2.1 and Theorem 3.1 that if K is a knot or link which has a diagram that is colorable mod n and K' is equivalent to K then every diagram of K' is colorable mod n. Note that the standard diagram of the unknot is not colorable mod n for $n > 1$. Hence, any knot that is colorable mod n for $n > 1$ is knotted.

Corollary 3.2. The trefoil knot is knotted.

Proof. Figure 9 shows that the trefoil knot has a diagram which is colorable mod 3. Thus, it is known that non–trivial knots exist.

A link L is *splittable* if there exists a polyhedral 3–ball B such that some of the components of L lie in the interior of B and the rest of the components lie in the complement of B. Thus, if L is splittable one can find a diagram D of L such that the images of the components that lie in the splitting ball B are disjoint from the images of the components that lie outside of B. By giving these disjoint images a monochrome (single integer) coloring (see Figure 10) we get the following:

Corollary 3.3. A split link is colorable mod 3.

This leads to the following amusing corollary.

Corollary 3.4. The Hopf link is not splittable.

Proof. The standard diagram of the Hopf link is not colorable mod 3. See Figure 11.

Note that a "generalized Hopf link" (see Figure 12) is not colorable mod n for any $n > 1$. Therefore we get the following corollary.

Corollary 3.5. If a link L is colorable mod n for $n \geq 2$, then L is not equivalent to a generalized Hopf link.

4. The Group of a Knot and Its Relationship to Colorability mod n. Let K be a knot (possibly wild or non–compact). The *group of* K is defined to be $\pi_1(\mathbb{R}^3 - K)$ (the group of homotopy classes of closed curves based at some point of $\mathbb{R}^3 - K$). A standard fact of knot theory is:

Theorem 4.1. Let K be a knot (possibly wild or non–compact). If K is unknotted, then the group of K is isomorphic to the infinite cyclic group Z.

Proof. See any standard reference on knot theory, e.g., Chapter 3 of reference [4].

Thus, a standard way to prove that a knot is knotted is to compute its group and show that its group is not isomorphic to $\mathbb{Z}$. The following theorem is known for tame knots but also applies to non–compact and to certain wild knots (Fox [3]).

Theorem 4.2. Let K be a tame knot (possibly non–compact) or link. If K is colorable mod n, then there is a homomorphism of the group of K onto the dihedral group D_n.

It follows from Theorem 4.2 that if a non–compact knot K is colorable mod n then K is knotted. We needed a theorem of this sort since there is no established set of Reidemeister moves for non–compact knots.

Proof. Recall that the Dihedral group D_n (which is often viewed as the group of reflections and rigid rotations of a regular polygon) has the following presentation:

$$< x, y \mid x^2 = 1 = y^n,\ xyx = y^{n-1} > .$$

Suppose K has a diagram D which has a crossing at which two distinct integers appear. Use the Wirtinger method (Chapter 3, section D of reference [4] or Chapter 10 of reference [7]) to obtain a presentation of the group of K. Let $\{c_1, c_2, \ldots, c_p\}$ be the Wirtinger generators of the group of K. Notice that, as one moves along D, a generator of the group K changes precisely when the integer assignment of a strand of D changes. Hence, we can assume that the strand associated with a generator c_i has "color" i (mod n). We can now obtain a function from the group of K to D_n in the following manner:

$$f(c_i^{\pm 1}) = xy^{i \text{ (mod } n)}.$$

f is onto because $\{x, xy\}$ generate D_n and are in the range of f.

Note that each crossing induces the following relation in the group of K (Figure 13):

$$c_j c_i c_j^{-1} = c_k.$$

It turns out that there are all of the relations in the group of K. But, because K is colorable mod n, we have that $k = 2j - 1 \mod n$.

To see that f is a group homomorphism:

$$f(c_j)f(c_i)f(c_j^{-1}) = xy^{j \pmod n} xy^{i \pmod n} xy^{j \pmod n}$$

$$= y^{n-j}y^i xy^j = x^{n+i-j}xy^j = y^{i-j}xy^j = xy^{n-(i-j)}y^j$$

$$= xy^{2j-i} = xy^k = f(c_k) = f(c_j c_i c_j^{-1}),$$

where mod n has been suppressed on the second and third lines.

Suppose now that each diagram D of K has no crossing where 2 distinct integers appear. It follows that D has at least two components D_1 and D_2 which come from components K_1 and K_2 of K. K_1 and K_2 can be split by either a polyhedral ball (if K is compact) or a polyhedral plane. The split "components" of the diagram can be colored by distinct integers, say 0 and 1.

Then one can obtain a map from the group of K to D_3 by mapping every generator of the group of K_1 to x and every generator of the group of K_2 to xy. This map is an onto group homomorphism. Note that it follows from the Seifert–Van Kampen Theorem that

$$\pi_1(\mathbb{R}^3 - K) \simeq \pi_1(\mathbb{R}^3 - K_1) * \pi_1(\mathbb{R}^3 - K_2).$$

(Here "$*$" denotes "free product".)

5. Applications. We now give a sample of applications of Theorem 4.2.

Example 5.1. Define a *k–tangle* T to be a polyhedral 3–ball B together with n disjoint properly embedded polyhedral arcs. Suppose T has a diagram that can be colored mod n for $n \geq 2$ such that the endpoints of the arcs of T have the same integer color. There is no way to connect the endpoints of T by polyhedral arcs outside of B so as to obtain either the unknot or a generalized Hopf link. To see this, just give the arcs outside of B the appropriate monochrome coloring. Thus, the knot or link so obtained is colorable mod n. If k is odd, then any noncompact knot containing T as a "sub–tangle" must be knotted for similar reasons. See Figure 14.

Example 5.2. Consider the non–compact knot in Figure 6. This is a non–compact knot version of the well known "Remarkable Simple Closed Curve". It is easy to see that this non–compact knot is colorable mod 3. Hence, this non–compact knot is knotted (though its diagram can be deformed into the diagram of an unknotted non–compact knot via a countable number of Reidemeister moves). The "knottedness" of the remarkable simple closed curve was first proved by Fox [6].

Example 5.3. Consider the non–compact knot K shown in Figure 15a. We can think of K as being formed by a countable union of 3–tangles glued together in a standard way. (See Figure 15b.) Each tangle can be thought of as an *element* of a *defining sequence* for K. Note that if we order the intersections of the arcs of the tangle with the end disks, which will be referred to as D^+ and D^-, the color assignment of the endpoints of the arcs which hit the end disk D^+ is a permutation of the color assignment of the endpoints of the arcs hitting D^-, namely (01). Therefore, it is clear that K is colorable mod n and therefore knotted.

Example 5.4. Figure 16a depicts a Fox–Artin [8] non–compact knot K_a and Figure 16b shows 3 consecutive elements in a defining sequence; we'll denote the "chunks" associated with the elements by N_1, N_2 and N_3 and index their end disks respectively. Note that the color assignment of the strands which hit D_3^+ is a permutation of the color assignment of the strands which hit D_1^-, namely (12). Thus, K_a is colorable mod 3 and therefore knotted. Note that we can replace the upper two strands which hit D_3^+ by any 2–tangle whatsoever (so long as the arcs of the tangle remain in the ball depicted in Figure 17) to obtain K_a'. It is easy to see that K_a' is colorable mod 3. In fact, the replacement tangle can be different for every N_j where $j = 0 \mod 3$. Similarly, we can replace any single colored strand by an appropriately knotted strand without affecting the colorability mod 3 property of K_a'. Hence, in such cases, it is easier to determine that K_a' is knotted by using coloring than it is to compute the group of K_a'.

Example 5.5. Figures 3 and 18 show wild knot analogs of the non–compact knots discussed in Examples 5.2 and 5.3. These wild knots have diagrams which can be colored mod 3; except, of course, for the images of the wild point. We assign no color to the image of the wild point. The following corollary follows from the limit group techniques discussed in reference [8] and Theorem 4.2.

Corollary 5.6. If K is a wild knot (possibly non–compact) with the following properties:

1) The set of wild points $p_1, p_2, \dots, p_k, \dots$ of K is a discrete set,

2) K has a diagram D such that one of the subarcs $\overline{p_i p_{i+1}}$ is colorable mod $n (n \geq 2)$ and

3) the subarcs of K between the wild points lie in 3–balls whose interiors are mutually disjoint and whose boundaries intersect only at the p_i.

Then K is knotted.

Figures 3 and 18 show situations in which Corollary 5.6 can be used. The reader is invited to find a way to either relax or remove condition 3.

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Figure 1.

Figure 2.

Figure 3.

Figure 4.

Figure 5.

Figure 6.

Figure 7.

Figure 8.

Figure 9.

Figure 10.

Figure 11.

Figure 12.

Figure 13.

Figure 14.

Figure 15a.

Figure 15b.

Figure 16a.

Figure 16b.

Figure 17.

Figure 18.