## SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

**47**. [1992, 88; 1993, 94; 1994, 99] *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Find all the solutions of

$$(x-1)x(x+1)(x+2) = -1.$$

*Comment by the editor.*

This problem was also solved by J. Sriskandarajah, University of Wisconsin Center-Richland, Richland Center, Wisconsin.

**61**. [1993, 131] *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Let $n$ be a positive integer greater than one. Prove that

$$(n-1)!\left(\sum_{k=1}^{n} k(k!)^{\frac{1}{k}}\right)\left(\prod_{k=1}^{n}(k!)^{\frac{1}{k}}\right) < 2^{n(n+3)/2}.$$

*Solution I by the proposer.*

From the relationship between the arithmetic and the geometric means of the positive integers $1, 2, 3, \ldots, k$, we have

$$(k!)^{\frac{1}{k}} < \frac{k+1}{2}.$$

This implies that

$$k(k!)^{\frac{1}{k}} < \frac{k(k+1)}{2}. \tag{1}$$

Next, from (1) we get

$$\sum_{k=1}^{n} k(k!)^{\frac{1}{k}} < \sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6} \tag{2}$$

and

$$\prod_{k=1}^{n} k(k!)^{\frac{1}{k}} = n! \prod_{k=1}^{n} (k!)^{\frac{1}{k}} < \prod_{k=1}^{n} \frac{k(k+1)}{2}. \tag{3}$$

But, by mathematical induction for $k > 1$, we can show that

$$\frac{k(k+1)}{2} < 2^k. \tag{4}$$

Hence, from (3) and (4) we deduce that

$$n! \prod_{k=1}^{n} (k!)^{\frac{1}{k}} < 2^{\frac{n(n+1)}{2}}. \tag{5}$$

Now, multiplying the corresponding sides of (2) and (5), and then dividing both sides of the resulting inequality by $n$, we obtain

$$(n-1)! \left( \sum_{k=1}^{n} k(k!)^{\frac{1}{k}} \right) \left( \prod_{k=1}^{n} (k!)^{\frac{1}{k}} \right) < \left( \frac{(n+1)(n+2)}{6} \right) 2^{\frac{n(n+1)}{2}}.$$

Again, from (4) we have

$$\frac{(n+1)(n+2)}{6} < 2^n.$$

Finally, the last two inequalities will give us the desired inequality.

*Solution II by N. J. Kuenzi and Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin (jointly).*

We will establish a slightly stronger inequality by showing that

$$(n-1)!\left(\sum_{k=1}^{n} k(k!)^{\frac{1}{k}}\right)\left(\prod_{k=1}^{n}(k!)^{\frac{1}{k}}\right) < 2^{n(n+1)/2} \quad \text{for} \quad n \geq 1.$$

The proof will be based on the following three inequalities:

$$(1) \qquad (k!)^{\frac{1}{k}} \leq \frac{k+1}{2} \quad \text{for} \quad k \geq 1,$$

$$(2) \qquad n(n+1) < 2^n \quad \text{for} \quad n \geq 5, \quad \text{and}$$

$$(3) \qquad \frac{(n+1)!(n+2)!}{3!} < 2^{n(n+3)/2} \quad \text{for} \quad n \geq 1.$$

The first inequality follows immediately from the arithmetic mean-geometric mean inequality,

$$(k!)^{1/k} \leq \frac{1+2+\cdots+k}{k} = \frac{k+1}{2}.$$

The second inequality can be established in a straightforward manner using mathematical induction. The third inequality can be numerically verified for $n = 1$, 2, and 3. Suppose the inequality holds for $n \geq 3$. Then

$$\frac{(n+2)(n+3)(n+1)!(n+2)!}{3!} < (n+2)(n+3)2^{n(n+3)/2}.$$

Using (2) yields

$$\frac{(n+2)(n+3)(n+1)!(n+2)!}{3!} < (2^{n+2})2^{n(n+3)/2}.$$

So,

$$\frac{(n+2)!(n+3)!}{3!} < 2^{(n+1)(n+4)/2}.$$

Thus, by induction and numerical verification, inequality (3) holds for $n \geq 1$. Next, using (1), we have

$$\sum_{k=1}^{n} k(k!)^{\frac{1}{k}} \leq \sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{3!}$$

and

$$\prod_{k=1}^{n} (k!)^{\frac{1}{k}} \leq \prod_{k=1}^{n} \frac{k+1}{2} = \frac{(n+1)!}{2^n}.$$

So,

$$\begin{aligned}(n-1)!\left(\sum_{k=1}^{n} k(k!)^{\frac{1}{k}}\right)\left(\prod_{k=1}^{n} (k!)^{\frac{1}{k}}\right) &\leq (n-1)!\left(\frac{n(n+1)(n+2)}{3!}\right)\frac{(n+1)!}{2^n} \\ &= \frac{(n+1)!(n+2)!}{3!2^n}.\end{aligned}$$

Using (3) yields

$$\frac{(n+1)!(n+2)!}{3!2^n} < \frac{2^{n(n+3)/2}}{2^n} = 2^{n(n+1)/2}.$$

**62**. [1993, 131] *Proposed by Jayanthi Ganapathy, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.*

Express explicitly in terms of $x$, all those functions $f(x)$ with domain $(M, \infty)$ for some real number $M$, that have the following properties.

(a) $f$ is increasing and differentiable on $(M, \infty)$.

(b) $0 < f'(x) < 1$ whenever $f(x) > 0$ and $f'(x) > 1$ whenever $f(x) < 0$.

(c)

$$e^{f(x)} - f'(x) = -\left(\frac{1}{e^{f(x)}} - \frac{1}{f'(x)}\right)$$

for all $x$ in $(M, \infty)$.

*Solution by N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.*

Suppose $f(x)$ is a function with domain $(M, \infty)$ which satisfies the three properties above. It follows from (c) that

$$e^{f(x)} - f'(x) = -\left(\frac{f'(x) - e^{f(x)}}{f'(x)e^{f(x)}}\right) = \frac{e^{f(x)} - f'(x)}{f'(x)e^{f(x)}}.$$

So,

$$(e^{f(x)})f'(x)[e^{f(x)} - f'(x)] = e^{f(x)} - f'(x).$$

Next, using (b), we have

$$e^{f(x)} - f'(x) > 1 - 1 = 0 \quad \text{whenever} \quad f(x) > 0 \quad \text{and}$$

$$e^{f(x)} - f'(x) < 1 - 1 = 0 \quad \text{whenever} \quad f(x) < 0.$$

So

$$(e^{f(x)})f'(x) = 1 \quad \text{whenever} \quad f(x) \neq 0.$$

If $x_0$ is a value such that $f(x_0) = 0$, then

$$f'(x_0)[1 - f'(x_0)] = 1 - f'(x_0) \quad \text{which implies} \quad f'(x_0) = 1.$$

Hence,

$$e^{f(x)}f'(x) = 1 \quad \text{for all } x \text{ in the domain.}$$

Taking anti-derivatives of each side of the last equation yields

$$e^{f(x)} = x - c \quad \text{for some constant} \quad c.$$

So,

$$f(x) = \ln(x - c) \text{ for all } x > M.$$

Note, $c$ can be any constant subject to the restriction that $c \leq M$. If $c < M$, then $f(x; c) = \ln(x - c)$ with domain restricted to $(M, \infty)$ satisfies the given properties.

*Also solved by Joseph E. Chance, University of Texas-Pan American, Edinburg, Texas; Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin and the proposer.*

*Comment by some readers.*

Joe Flowers, Northeast Missouri State University, Kirksville, Missouri and Kandasamy Muthuvel noted that the assumption that $f$ is increasing is redundant since it follows from parts (b) and (c) of the problem and an argument given in the featured solution to the problem.

**63**. [1993, 132] *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Find the roots of the equation

$$8x^6 - 42ix^5 - 21x^4 - 84ix^3 - 21x^2 - 42ix + 8 = 0,$$

where $i^2 = -1$.

*Solution I by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin and J. Sriskandarajah, University of Wisconsin Center-Richland, Richland Center, Wisconsin.*

$$\begin{aligned}
&8x^6 - 42ix^5 - 21x^4 - 84ix^3 - 21x^2 - 42ix + 8 \\
&= (x^2+1)(8x^4 - 29x^2 + 8 - 42ix^3 - 42ix) \\
&= (x^2+1)[8(x^2+1)^2 - 42ix(x^2+1) - 45x^2] \\
&= (x^2+1)[4(x^2+1) - 15ix][2(x^2+1) - 3ix] \\
&= (x^2+1) \cdot 4\left(x^2 + 1 - \frac{15}{4}ix\right) \cdot 2\left(x^2 + 1 - \frac{3}{2}ix\right) \\
&= 8(x-i)(x+i)(x-4i)\left(x + \frac{1}{4}i\right)(x-2i)\left(x + \frac{1}{2}i\right).
\end{aligned}$$

Therefore, the desired roots are $i, -i, 4i, -\frac{1}{4}i, 2i,$ and $-\frac{1}{2}i$.

*Solution II by Dale Woods, Reeds Spring, Missouri; Leon Hall, University of Missouri-Rolla, Rolla, Missouri; Joseph E. Chance, University of Texas-Pan American, Edinburg, Texas; Mangho Ahuja, Southeast Missouri State University, Cape Girardeau, Missouri; Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri and the proposer.*

Let $x = iy$. Then the equation becomes (after multiplying by $-1$)

$$8y^6 - 42y^5 + 21y^4 + 84y^3 - 21y^2 - 42y - 8 = 0.$$

Using synthetic division we obtain

$$y = 1, -1, 2, 4, -\frac{1}{2}, -\frac{1}{4}.$$

Therefore, the roots of the original equation are

$$x = i, -i, 2i, 4i, -\frac{i}{2}, -\frac{i}{4}.$$

*Solution III by Herta T. Freitag, Roanoke, Virginia and the proposer.*

Assuming $x \neq 0$, we divide by $x^3$ to obtain

$$8\left(x^3 + \frac{1}{x^3}\right) - 42i\left(x^2 + \frac{1}{x^2}\right) - 21\left(x + \frac{1}{x}\right) - 84i = 0.$$

Now we let

$$x + \frac{1}{x} = y.$$

Then

$$x^2 + \frac{1}{x^2} = y^2 - 2 \quad \text{and} \quad x^3 + \frac{1}{x^3} = y^3 - 3y.$$

Thus, our equation becomes

$$8(y^3 - 3y) - 42i(y^2 - 2) - 21y - 84i = 0$$

or

$$8y^3 - 42iy^2 - 45y = 0.$$

Therefore, $y_1 = 0$ and $y_2$ and $y_3$ are the solutions of

$$8y^2 - 42iy - 45 = 0.$$

It follows that

$$y_{2,3} = \frac{21i \pm 9i}{8}.$$

Therefore, we have

$$y_1 = 0, \quad y_2 = \frac{15i}{4}, \quad \text{and} \quad y_3 = \frac{3i}{2}.$$

But

$$y = x + \frac{1}{x}.$$

Thus

$$x + \frac{1}{x} = 0,$$

from which $x^2 + 1 = 0$ and $x_{1,2} = \pm i$. Also,

$$x + \frac{1}{x} = \frac{15i}{4},$$

or $4x^2 - 15ix + 4 = 0$, from which

$$x_{3,4} = \frac{15i \pm 17i}{8}.$$

Hence, $x_3 = 4i$ and $x_4 = -i/4$. Finally, from

$$x + \frac{1}{x} = \frac{3i}{2},$$

that is, $2x^2 - 3ix + 2 = 0$, we have

$$x_{5,6} = \frac{3i \pm 5i}{4},$$

i.e., $x_5 = 2i$ and $x_6 = -i/2$. Therefore, the solutions are $i, -i, 4i, -i/4, 2i, -i/2$.

*Also solved by Leonard L. Palmer, Southeast Missouri State University, Cape Girardeau, Missouri and Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico.*

*Comment by some readers.*

Bob Prielipp noted that the desired roots can be found using the POLY feature of the TI-85 calculator. One must remember to enter $-42i$ as $(0, -42)$, for example. The calculator does essentially all of the rest of the work, however.

Leon Hall made the substitution in Solution II and then used *Mathematica* to solve the resulting equation. He notes that *Mathematica* does not handle the original equation very well.

**64**. [1993, 132] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

For $n$ a positive integer, let $M_n$ denote the $n \times n$ matrix $(a_{ij})$ where $a_{ij} = i + j$. Is there a simple formula for perm $(M_n)$?

*Comment by the proposer.*

I have no solution to this problem.

There are many neat formulas for the determinant of an interesting matrix, but almost no results for permanents. I have generated lots of data for the permanents of various simply-formed matrices, but have not been able to come up with any pattern in the resulting numbers.

Here's some numerical data:

| $n$ | perm $(M_n)$ | factorization |
|---|---|---|
| 1 | 2 | 2 |
| 2 | 17 | 17 |
| 3 | 336 | $2^4$ 3 7 |
| 4 | 12052 | $2^2$ 23 131 |
| 5 | 685080 | $2^3$ $3^2$ 5 11 173 |
| 6 | 56658660 | $2^2$ 3 5 23 41057 |
| 7 | 6428352000 | $2^9$ 3 $5^3$ 7 4783 |
| 8 | 958532774976 | $2^6$ $3^3$ 7 31 67 38153 |
| 9 | 181800011433600 | $2^7$ $3^5$ $5^2$ 7 19 $23^2$ 3323 |

*Comment by the editor.*

No solution to this problem has been received. It therefore remains open.

The editor submitted the first 9 terms of this sequence to *The On-Line Encyclopedia of Integer Sequences,* by N. J. A. Sloane of AT&T Bell Labs, Murray Hill, New Jersey (with the assistance of Simon Plouffe of the Universite' du Quebec a' Montreal). To lookup this sequence in the Encyclopedia, I sent mail to sequences@research.att.com containing a line of the form

lookup 2 17 336 12052 685080 56658660 6428352000 958532774976 181800011433600

However, no match of this sequence was found in the Encyclopedia. Later, I discovered that there is a second program to identify sequences. This one will not just look up the sequence in the Encyclopedia, it will also try a large number of tricks in order to attempt to explain the sequence. I sent a message to superseeker@research.att.com containing the line

lookup 2 17 336 12052 685080 56658660 6428352000 958532774976 181800011433600

In addition to trying to look up the sequence in the Encyclopedia, the superseeker will use the "gfun" Maple package of Bruno Salvy and Paul Zimmerman and the "findhard" program of Simon Plouffe and Bruno Salvy. Again however, none of the tests led to an explanation of this sequence.