AN HISTORICAL CONTRADICTION

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1. Introduction. One of the most intriguing aspects of mathematics is the discovery of a relationship between two seemingly unrelated branches. One such relationship occurs in the criteria for construction of regular polygons using only Euclidean tools. The question, inherited from the studies of the ancient Greeks, may be phrased, "what types of regular polygons may be constructed, using only the unmarked ruler and compass?" Though the ancients were able to make some progress into this inquiry, this question wasn't completely resolved until some 2000 years later.

The ancient Greeks were able to establish that regular polygons of 3, 5, and 15 sides were constructible. By the process of inscribing these polygons in circles and repeatedly bisecting the sides, regular polygons of $3(2^m)$, $5(2^m)$ or $15(2^m)$ sides may be obtained, where m is the number of bisections. In the early nineteenth century Carl Friedrich Gauss made the first addition to this theory by showing the steps whereby a regular heptadecagon (17–sides) can be constructed using only straight–edge and compass. He was also able to show in his *Disquisitions Arithmeticae* that a regular polygon of p–sides was constructible if p is a Fermat prime or a product of distinct Fermat primes (or the same multiplied by a power of 2).

Let us recall that a Fermat number is a number of the form $2^r + 1$. There are only five numbers of this form that are known to be prime. In fact, they are the first five Fermat numbers: 3, 5, 17, 257 and 65,537. It seems remarkable that a purely theoretical classification of a prime number type should play a role in the classification of polygon construction. The fact that Fermat died about 100 years before Gauss was born is just as fascinating. Another feature of this breakthrough was the vast amount of time between the initial discoveries and the "final" dismissal of the problem.

2. The Historical Question. Was the Gaussian criteria for polygon construction complete? Though many historians of mathematics credit Gauss for the complete solution, we will find upon further inspection that this was probably not the case. It is true that Gauss proves in the *Disquisitions* that p being a Fermat prime or product of distinct Fermat primes is a sufficient condition for the construction of a regular p–sided polygon. To determine whether these are the **only** ones that can be constructed, let us consider Gauss's treatment in his *Disquisitions*:

> "(art. 365) Whenever $n - 1$ implies prime factors other than 2, we are always led to equations of higher degree, namely, to one or more cubic equations when 3 appears once or several times among the prime factors of $n - 1$, to equations of the fifth degree when $n - 1$ is divisible by 5, etc. WE CAN SHOW WITH ALL RIGOR THAT THESE HIGHER–DEGREE EQUATIONS CANNOT BE AVOIDED IN ANY WAY NOR CAN THEY BE REDUCED TO LOWER–DEGREE EQUATIONS. The limits of the present work exclude the demonstration here, but we issue this warning lest anyone attempt to achieve geometric constructions for sections other than the ones suggested by our theory (e.g. sections into 7, 11, 13, 19, etc. parts) and so spend his time uselessly."

> "(art. 366) ... in order to be able to divide the circle geometrically into N parts, ... it is required that N imply no odd prime factor that is not of the form $2^m + 1$ nor any prime factor of the form $2^m + 1$ more than once."

In article 365, Gauss states that the *Disquisitions* excludes the demonstration of the requirement of necessity. Article 366 states that if a constructible polygon has N sides, then every odd prime factor of N is a Fermat prime or product of distinct Fermat primes. No proof follows this last statement of Gauss's book. It seems clear that Gauss makes the conjecture that the conditions he previously proved as sufficient were also necessary for the construction of regular polygons. **However, he does not follow this with a proof that these are the only polygons that are constructible.** N. D. Kazarinoff points out that, "Gauss never published a proof of this assertion, nor did he ever outline one in his correspondence or notes."

It would be a logical fallacy to say that just because we do not have a published proof of his statement, that Gauss did not have one. Mathematicians are usually very precise and clear not to give credit for establishing an idea until it is accompanied by a proof. The most obvious example would be Pierre de Fermat's marginal note in his copy of Diophantus' *Arithmetica* that he had found a proof that the equation $x^n + y^n = z^n$ has no solution for $n \geq 3$. The mathematical community is very quick to mention the fact that no proof from Fermat has ever been found. In fact, it has taken the finest mathematicians over 300 years to prove that Fermat's conjecture is indeed true.

It would seem very odd then, to give Gauss credit for establishing the necessary and sufficient conditions of regular polygon construction. However, much of the mathematical literature on this subject does just this. Consider:

1. From Ribenboim (1991):

> " ... with the famous result of Gauss (see *Disquisitions Arithmeticae* articles 365, 366 – the last ones in the book – as a crowning result for much of the theory previously developed). He showed that if $n \geq 3$ is an integer, the regular polygon with n sides may be constructed by ruler and compass, if and only if $n = 2^k p_1 p_2 \cdots p_h$, where $k \geq 0$, $h \geq 0$ and $p_1, \ldots,$ p_h are distinct odd primes, each being a Fermat number."

2. From Eves (1990):

> "In 1796, the eminent German mathematician Carl Frederick Gauss developed the theory that showed that a regular polygon having a prime number of sides can be constructed with Euclidean tools if and only if that number is of the form $f(n) = 2^{2^n} + 1$."

3. From Burton (1976):

> "In the seventh and last section of the *Disquisitions Arithmeticae*, Gauss proved that a regular polygon of n sides is so constructible if and only [if] either
>
> $$n = 2^k \quad \text{or} \quad n = 2^k p_1 p_2 \cdots p_r,$$
>
> where $k \geq 0$ and $p_1, p_2, \ldots, p_r$ are distinct Fermat primes."

4. From Coxeter (1961):

> "This question was completely answered by Gauss (1777–1855) at the age of nineteen. Gauss found that a regular n–gon, say $\{n\}$, can be so constructed if and only if the odd prime factors of n are distinct "Fermat primes" ... "

5. From E. T. Bell (1937):

> "The young man proved that a straight–edge and compass construction of a regular polygon having an odd number of sides is possible when, and only when, that number is either a prime Fermat number (that is a prime of the form $2^{2^n} + 1$), or is made up by multiplying together different Fermat primes ... His name was Gauss."

The apocryphal nature of quotations 3, 4, and 5 were first brought to my attention by Francis (1985). The first two quotations show the relevance of this issue today. Similar quotes can be found in many other texts, including Klein and Boyer.

We have just seen statements saying that Gauss 'completely answered', 'showed', or 'proved' the necessary and sufficient conditions for regular polygon construction. These statements occur without justification. In light of our previous discussion, we see that these statements are impossible to justify.

If Gauss did indeed have a proof of the necessary and sufficient conditions of the constructibility criteria, all indications tell us that he would have made it known publicly. An immediate result that follows from the criteria is that a regular nonagon (9–sided) is not constructible since 9 is the product of two *like* Fermat primes, 3. In this case the central angle of 40 degrees would not be constructible. We know, however, that the 120 degree angle is easily constructed. This trivial example would prove that there is no general method for trisecting an angle.

Certainly, a mathematician of Gauss's abilities would have realized the above example. The angle trisection problem was one of the three famous problems of antiquity, receiving an incredible amount of attention. If Gauss had a proof of this, he would have wanted to receive credit for it. Let us digress for a moment to the case of the regular heptadecagon. Gauss was so proud of this accomplishment that he requested that the figure be inscribed on his tombstone. It only seems reasonable that if Gauss could have dismissed the possibility of trisecting the angle, he would have made it known.

To make matters even more interesting, historians usually credit the solution of the angle trisection problem to Pierre Wantzel in 1837. This is 41 years after Gauss's discoveries. Several examples can be cited:

1. From Yates (1942):

> "P. L. Wantzel in 1837 was the first to give a rigorous proof of the impossibility of trisecting the general angle by straightedge and compass."

2. From Bell (1945):

> "These two [trisection and duplication] were not settled till P. L. Wantzel (A.D. 1814–1848, French) in 1837 obtained necessary and sufficient conditions .... Thus the problems were proved impossible."

3. From Eves (1990):

> "1837–trisection of an angle and duplication of cube proved impossible. (Chronological Table)"

Other similar quotes may be found in Kline, Jones, Greenberg, and Kazarinoff (among many others). All this evidence tells us that it was highly unlikely that Gauss had a proof of the necessity of the aforementioned conditions.

3. Who Deserves Credit. Credit for the *conjecture* which later turned out to be true should be given to Gauss (1796). Considerable admiration should be given to Gauss for his discoveries. However, credit for the *proof* of the necessity of the constructibility condition should be given to Wantzel (1837). He gives the first known proof of the necessity of the conditions. In fact, the proof occurs in the same article in which he shows the impossibility of trisecting the angle. However, it is rare in the literature to see this credit given.

Some historians give credit to people besides Wantzel for this discovery. Consider Archibald (1914),: "In the first part of his paper Professor Pierpont shows 'that the condition which Gauss gave as necessary is in fact such.' " Also see Pierpont.

A more common treatment is to give Gauss credit for the sufficiency statement only. Often no credit is given to anyone when discussing the necessity of the theorem. Consider:

1. From *Encyclopedia Britannica* (1992):

> "Gauss's remarkable achievement was to show (at age 18) that it is possible with ruler and compass to construct a regular n–gon if n is either a prime Fermat number or a product of different prime Fermat numbers ... It was subsequently shown that the converse of Gauss's theorem holds and that, therefore these are the only regular polygons that can be so constructed."

2. From Apostol (1976):

> "Gauss proved that if $F_n = p$, then a regular polygon of p sides can be constructed with straightedge and compass."

Similar quotes may be found in Dudley, Dunham, or Scott.

4. Conclusion. It seems proper to give credit to Gauss for making the conjecture that a regular polygon of p–sides may be constructed if and only if p is a Fermat prime or a distinct product of Fermat primes. Gauss also deserves credit for proving the sufficiency of the theorem. Pierre Wantzel deserves the credit for first demonstrating the necessity of the criteria.

As mathematicians we pride ourselves on exactness and preciseness. It is amazing that with the attention paid to detail, that such an incredible discovery was not given its proper attention. One possible reason for this dilemma is that noted mathematicians made errors (i.e. Klein, Bell, or Coxeter). Their point of view was accepted as fact by many non–investigative people. As Francis puts it, "Rumor, reinforced by many years of acceptance and exaggeration, dies hard."

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