SEQUENCES OF MAPPINGS AND THE EXPANSIVE PROPERTY

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1. Introduction. Consider a sequence, $\{f_n\}$, of continuous functions of a metric space, (X, d), onto another metric space, (Y, d'), that converges pointwise to another continuous function, f, from X to Y. That is, for each x in X, the sequence $\{f_n(x)\}$ converges to $f(x)$. Let us further suppose that for each n, f_n has property P. It is an interesting problem to see which properties P also hold for f. For certain properties, such as continuity and differentiability, this is a standard question in undergraduate real analysis.

2. Examples. Let f_n be a sequence of uniformly continuous functions on $[a, b]$ that converges uniformly to another function, f. If, given $\epsilon > 0$, the same δ works for all n, it is easily seen that f is uniformly continuous. Uniform continuity is therefore preserved.

Let f_n be a sequence of nonincreasing (or nondecreasing) functions. If f_n converges to f, then f is nonincreasing (or nondecreasing). Therefore the property of being nonincreasing (or nondecreasing) is preserved.

The property of being increasing (or decreasing) is not preserved. To see this let $f_n(x) = x/n$ (or $-x/n$) and $f(x) = 0$.

To consider continuity, let

$$f_n(x) = \begin{cases} -1, & \text{if } x < -1/n; \\ nx, & \text{if } -1/n \leq x \leq 1/n; \\ 1, & \text{if } x > 1/n \end{cases}$$

and

$$f(x) = \begin{cases} -1, & \text{if } x < 0; \\ 0, & \text{if } x = 0; \\ 1, & \text{if } x > 0. \end{cases}$$

We see then, that continuity is not preserved under convergence.

To consider differentiability, let

$$f_n(x) = \begin{cases} -2x - 1/n, & \text{if } x < -1/n; \\ nx^2, & \text{if } -1/n \le x \le 1/n; \\ 2x - 1/n, & \text{if } x > 1/n \end{cases}$$

and

$$f(x) = |2x|.$$

The sequence converges pointwise to f so we see that differentiability is not preserved.

3. Expansiveness. Let $\{f_n\}$ be a sequence of continuous functions that converges to another continuous function, f. Suppose under repeated iterations, each f_n takes any pair of distinct points "far apart". Does f always take distinct pairs of points "far apart"? Suppose each f_n has a pair of points that is always kept "close". Is there a pair of points that is always kept "close" under repeated iterations of f?

The settings we will consider for our results are the real line, $\mathbb{R}$, and the unit circle, S^1. It should be noted, however, that many of the results we will consider are true in more general settings.

In 1950, W. R. Utz, who spent time at the University of Missouri campuses at Columbia and Rolla, defined what he called "unstable homeomorphisms" [2]. According to Utz, a homeomorphism, f, of a metric space, X, onto itself is called *unstable* if there is a positive number, ϵ, such that if x and y are distinct points of X, then there exists an integer, $n = n(x, y)$, such that $d(f^n(x), f^n(y)) > \epsilon$. It should be noted that n may be negative. In 1955, Gottschalk and Hedlund suggested the term *expansive homeomorphism* in place of unstable homeomorphism [1]. While "expansive" has come to be accepted for this concept, it seems that "unstable" is a more descriptive term. In personal conversation with the author, Utz explained his choice of the term "unstable." Rather than always moving points farther apart, such a homeomorphism may move points alternately farther apart and then closer together. Another advantage is that in other areas, particularly fixed point theory, the term "expansive" is used to mean a function that always moves points farther apart. Also, see the paragraph following Example 3, below, for another result that is somewhat counter–intuitive to the idea of "expansiveness." However, we will bow to convention and use the term "expansive".

In [4], R. K. Williams gives a good summary of many of the basic results on expansive homeomorphisms.

4. Definitions and Basic Results. By a *mapping*, we will mean a continuous function. A point, x, is a *fixed point* of f, if $f(x) = x$. Let n be a positive integer. Then f^0 is the identity function, $f^n = f(f^{n-1})$, and $f^{-n} = (f^{-1})^n$.

In [5], Williams generalized the concept of expansiveness to any mapping. Let f be a map of a metric space onto itself. Let x be any point of X. The *orbit* of x under f is defined by

$$O(x) = \bigcup_{n=-\infty}^{\infty} f^n(x).$$

A *suborbit* of x is a doubly–infinite sequence, $\{x_n\}_{n=-\infty}^{\infty}$, such that $x_0 = x$ and $f(x_i) = x_{i-1}$ for every integer, i. We say that f is an *expansive mapping*, with *expansive constant* ϵ, if there exists a positive number, ϵ, such that for any distinct points of X, x and y, and any suborbits, $\{x_n\}$ and $\{y_n\}$, of x and y, there exists an integer, N, dependent on x, y and the choice of suborbits, such that $d(x_N, y_N) > \epsilon$. Notice that if ϵ is an expansive constant for f, then any positive number less than ϵ will also be an expansive constant for f. Though this definition is fairly technical, it essentially says that either the points will be moved "far apart" or they came from points that were "far apart."

The following is a well known result that will be very useful in Examples 1 and 2 below.

Theorem. Let f be a mapping of a compact metric space onto itself. If the set of fixed points of f is infinite, then f is not expansive.

Proof. Let $\{p_n\}$ be a convergent sequence of distinct fixed points of f. Let ϵ be a positive number. Then there exist positive integers, m and k, such that $d(p_m, p_k) < \epsilon$. Then $\{\ldots, p_m, p_m, \ldots\}$ and $\{\ldots, p_k, p_k, \ldots\}$ are suborbits, $\{x_n\}$ and $\{y_n\}$, of p_m and p_k, respectively, for which $d(x_n, y_n) < \epsilon$ for every n. Thus, f is not expansive.

We first construct two sequences of expansive mappings, each of which converges to a mapping that is not expansive. The first is on the reals, the second on the unit circle, S^1. Note that all mappings on the unit circle will be defined in terms of their effect on the central angle. For our metric on S^1, with $x \leq y$, we use $d(x, y) = \min\{y - x, 2\pi + x - y\}$.

Example 1. Let $g_n: \mathbb{R} \to \mathbb{R}$ be defined by $g_n(x) = (1 + 1/n)x$. Now, $|g_n^k(x) - g_n^k(y)| = (1 + 1/n)^k |x - y|$. Thus, each g_n is expansive. However, g_n converges to g where $g(x) = x$. Since the set of fixed points of g restricted to any compact subset of $\mathbb{R}$ is infinite we have that g is not expansive.

We would like to find a sequence of expansive maps of a compact space that converges to a map that is not expansive. It would seem likely that if such a sequence were to exist that the sequence of expansive constants would converge to 0. As we shall see, this is not the case.

Example 2. Let $f_n: S^1 \to S^1$ be defined by

$$f_n(x) = \begin{cases} (1 + 1/(2n))x, & \text{if } 0 \leq x \leq \pi; \\ (3 - 1/(2n))(x - 2\pi) \pmod{2\pi}, & \text{if } \pi \leq x \leq 2\pi \end{cases}$$

(see Figure 1).

Claim. Each f_n is expansive with expansive constant $\pi/180$. (Note – We are making no claim that this is the largest possible expansive constant. It just happened to be one of the author's favorite numbers.)

Proof of Claim. Let x and y be elements of S^1 such that $0 < d(x, y) = \alpha < \pi/180$. We first show that $d(f_n(x), f_n(y)) \geq (1 + 1/(2n))\alpha$. There are four cases.

Case I. $0 \leq x, y \leq \pi$.

Case II. $\pi - \pi/180 \leq x \leq \pi < y \leq \pi + \pi/180$.

Case III. $\pi \leq x < y \leq 2\pi$.

Case IV. $0 \leq x \leq \pi/180$, $y \geq 2\pi - \pi/180$.

We will do only the proof for Case II. The others are similar.

Proof for Case II. $\pi - \pi/180 \leq x \leq \pi < y \leq \pi + \pi/180$.
Since $f(\pi)$ is between $f(x)$ and $f(y)$,

$$\begin{aligned}
d(f_n(x), f_n(y)) &= d(f_n(x), f_n(\pi)) + d(f_n(\pi), f_n(y)) \\
&= |(1 + 1/(2n))\pi - (1 + 1/(2n))x| \\
&\quad + |(3 - 1/(2n))y - (3 - 1/(2n))\pi| \\
&= (1 + 1/(2n))(\pi - x) + (3 - 1/(2n))(y - \pi) \\
&\geq (1 + 1/(2n))(\pi - x) + (1 + 1/(2n))(y - \pi) \\
&= (1 + 1/(2n))(y - x) \\
&= (1 + 1/(2n))\alpha.
\end{aligned}$$

Therefore, for x and y with $d(x, y) = \alpha < \pi/180$, $d(f_n(x), f_n(y)) \geq (1 + 1/(2n))\alpha$. There exists a positive integer k such that $(1 + 1/(2n))^k \alpha \geq \pi/180$. Thus, $d(f_n^k(x), f_n^k(y)) \geq \pi/180$. We then have that, for each n, f_n is expansive with expansive constant $\pi/180$.

Now, the sequence $\{f_n\}$ converges to $f(x)$, (see Figure 2), defined below.

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq \pi; \\ 3(x - 2\pi) \pmod{2\pi}, & \text{if } \pi \leq x \leq 2\pi. \end{cases}$$

Since S^1 is compact, the fact that f has infinitely many fixed points implies that f is not expansive.

We now turn to the problem of defining a sequence of mappings that are not expansive that converges to a mapping that is expansive. An example, again on S^1, with the desired property was given in [6]. We give a simpler example, also on S^1.

Example 3. That $F(x) = 2x$ is expansive follows from the fact that when two distinct points, x and y, are "close", i.e., if $d(x, y) < \pi/2$, the distance between them is doubled. This was first proved in [3] by R. F. Williams.

Let

$$F_n(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1/n; \\ \frac{4n\pi - 1}{2n\pi - 1}(x - 1/n) + \frac{1}{n}, & \text{if } 1/n \leq x \leq 2\pi \end{cases}$$

(see Figure 3).

Since the set of fixed points for $F_n(x)$ is infinite, we know that F_n is not expansive. However, F_n converges to F, given by

$$F(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \pi; \\ 2(x - 2\pi) \pmod{2\pi}, & \text{if } \pi \leq x \leq 2\pi \end{cases}$$

(see Figure 4), which is expansive.

Consider the following interesting fact about the function in Example 3. Let $x = p\pi/q$ be a point on S^1 where p is a positive integer and q is a positive integer power of 2. Then $F^k(x) = 0$ for all k such that $2^k > q$. Also,

$$\{p\pi/q \in S^1 \mid p \in \mathbb{Z}^+,\ q = 2^n,\ n \in \mathbb{Z}^+\}$$

is dense in S^1. Therefore the function is expansive yet there is a dense subset in which all points eventually map to (and henceforth stay at) 0.

An example similar to Example 3 (though without the dense subset going to 0) can be constructed on $\mathbb{R}$ using virtually the same function by using the second part of the definition from $1/n$ to ∞ and on $(-\infty, 0)$ letting $F_n(x) = 2x$.

All of these examples serve to underscore the fact that functions that are "almost the same" can have very different properties.

References

1. W. H. Gottschalk and G. A. Hedlund, *Topological Dynamics*, Amer. Math. Soc. Colloq. Publ., Vol. 36, Amer. Math. Soc., Providence, RI, 1955.

2. W. R. Utz, "Unstable Homeomorphisms," *Proceedings of the AMS*, 1 (1950), 769–774.

3. R. F. Williams, "A Note on Unstable Homeomorphisms," *Proceedings of the AMS*, 6 (1955), 308–309.

4. R. K. Williams, "The Elusive Expansive Property," *American Mathematical Monthly*, 86 (1979), 483–484.

5. R. K. Williams, "Expansive Mappings," *American Mathematical Monthly*, 75 (1968), 522–523.

6. F. E. Worth, "Concerning the Expansive Property and Shift Homeomorphisms of Inverse Limits," Dissertation, University of Missouri–Rolla, Rolla, MO, 1991.

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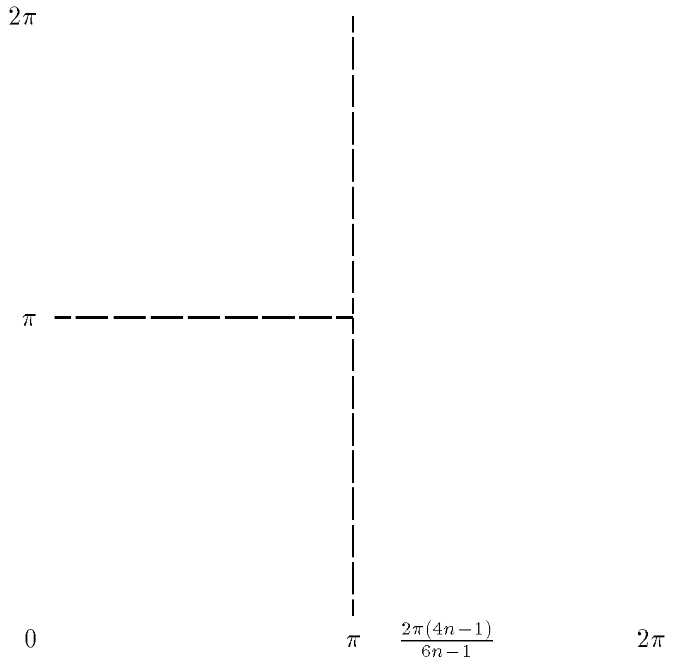

Figure 1
$f_n(x)$

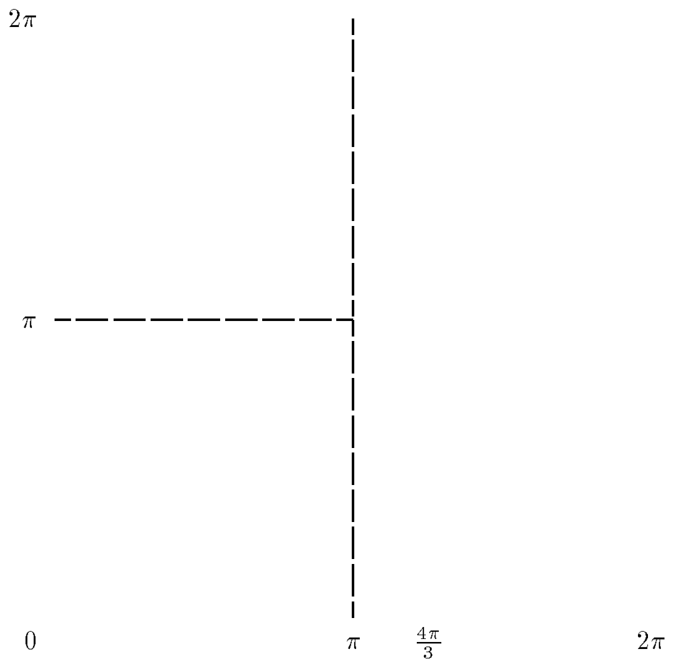

Figure 2
$f(x)$

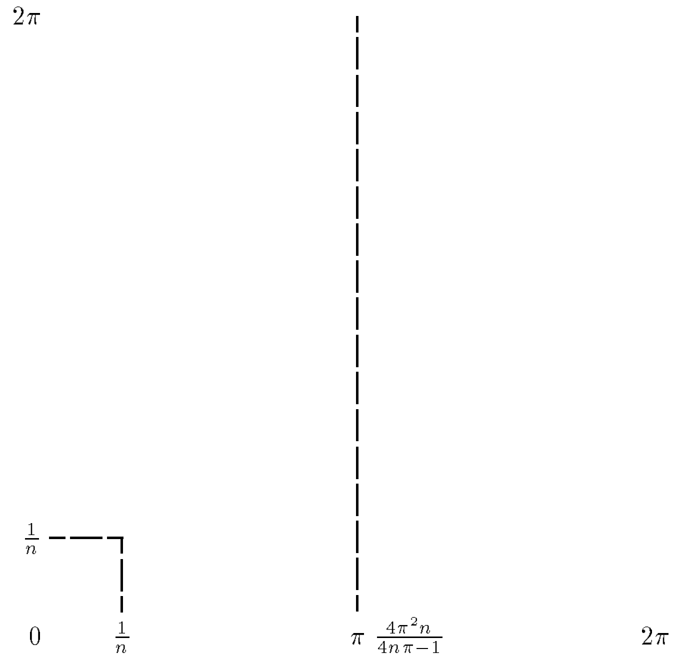

Figure 3
$F_n(x)$

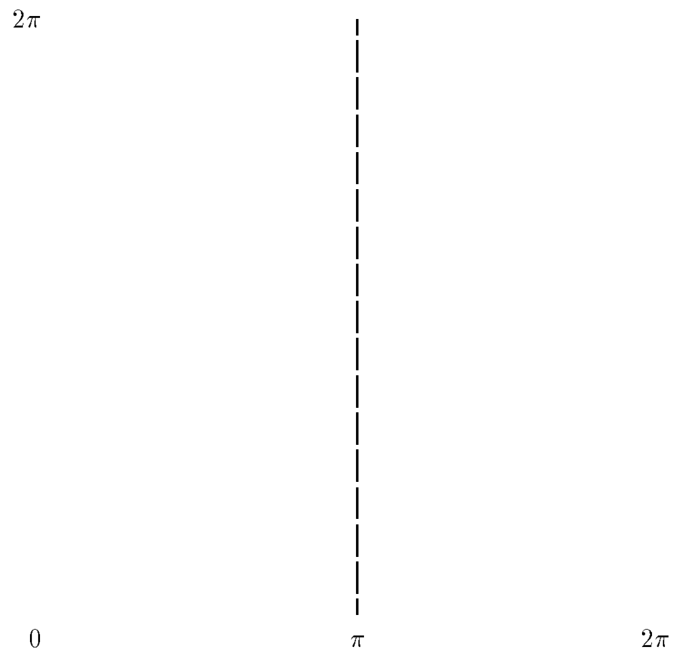

Figure 4
$F(x)$